THE THIRTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE thirteenth annual meeting of the Society was held in New York City on Friday and Saturday, December 28–29, 1906, forming a part of the general gathering of scientists in attendance at the meetings of the American Association for the Advancement of Science and the numerous affiliated societies. A noticeable effect of this reinforcement of interest was an enlarged and more widely representative attendance, considerably exceeding that of any previous meeting of the Society. Friday morning was devoted to a joint session with Section A of the Association and the Astronomical and Astrophysical Society. Professor Simon Newcomb presided; seven papers selected from the programmes of the participating organizations were presented before a large audience. The remaining three sessions were taken up with the regular programme, to which were also added several papers from Section A. Unfortunately the limited time did not permit of adequate presentation and discussion of even the most important papers announced, several being finally read by title. The productive capacity of the Society is constantly outrunning the facilities for presentation; a stricter enforcement of the rules regarding the early submission of titles and abstracts of papers, and a rigorous observance of the order of presentation announced on the printed programme are demanded to relieve the congestion and preserve the orderly conduct of the meetings. It has also been strongly urged that to promote intelligent discussion of the papers abstracts should be included in the printed programmes. This would certainly be a desirable feature. Its practical working out would require some increase and redistribution of administrative forces and responsibilities. The whole subject will receive the serious consideration of the Council.

The attendance at the several sessions of the annual meeting included the following eighty-one members of the Society:

Mr. William Betz, Mr. W. C. Brenke, Professor E. W. Brown, Professor W. G. Bullard, Dr. W. H. Bussey, Dr. W. B. Carver, Dr. J. E. Clarke, Professor F. N. Cole, Miss E. B.

Cowley, Dr. H. N. Davis, Dr. W. S. Dennett, Professor T. W. Edmondson, Professor F. C. Ferry, Professor J. C. Fields, Professor W. B. Fite, Professor A. S. Gale, Dr. F. L. Griffin, Miss Ida Griffiths, Professor G. B. Halsted, Professor Harris Hancock, Professor J. G. Hardy, Professor H. E. Hawkes, Professor E. R. Hedrick, Professor C. S. Howe, Professor L. S. Hulburt, Professor E. V. Huntington, Professor J. I. Hutchinson, Professor Harold Jacoby, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor W. W. Landis, Dr. G. H. Ling, Mr. L. L. Locke, Dr. W. R. Longley, Professor T. E. McKinney, Professor J. McMahon, Professor James Maclay, Dr. Isabel Maddison, Professor H. P. Manning, Professor Max Mason, Professor Mansfield Merriman, Professor W. H. Metzler, Professor H. B. Mitchell, Dr. C. L. E. Moore, Dr. R. L. Moore, Professor Frank Morley, Professor Richard Morris, Professor Simon Newcomb, Professor G. D. Olds, Professor James Pierpont, Professor R. W. Prentiss, Miss Amy Rayson, Mr. H. W. Reddick, Miss S. F. Richardson, Professor E. D. Roe, Miss Ida M. Schottenfels, Professor Arthur Schultze, Mr. E. I. Shepard, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. R. P. Stephens, Professor Ormond Stone, Professor W. E. Story, Dr. W. M. Strong, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor A. G. Webster, Professor L. G. Weld, Professor H. S. White, Professor E. B. Wilson, President R. S. Woodward.

Professor E. B. Van Vleck presided at the afternoon session on Friday, being relieved by Professor Morley. President H. S. White, Professor Morley, and Vice-President P. F. Smith occupied the chair at the Saturday sessions. The Council announced the election of Mr. E. I. Shepard, of Harvard University, to membership in the Society. Eight applications for membership were received.

Owing to recent illness, President Osgood was unable to deliver his address; it will be presented at the April meeting of the Society.

The organization of a new Section, to be known as the Southwestern Section of the Society, was authorized by the Council. It was decided to hold the next summer meeting at Cornell

University. The next annual meeting was fixed for Friday and Saturday, December 27–28, 1907. An amendment to the constitution was adopted by which the Editorial Committee of the *Transactions* are included in the membership of the Council.

Reports were submitted by the Treasurer, Librarian, and Auditing Committee. These reports will appear in the forthcoming Annual Register. The membership of the Society has increased during the year from 504 to 548, including at present 50 life members. The number of papers presented at all meetings during the year, including the preliminary meeting of the Southwestern Section was 176, as against 147 in 1905. The total attendance of members was 350; 192 members attended at least one meeting during the year. The library now contains about 2,500 bound volumes; a list of the journals and of the accessions during 1905–1906 is printed in the Annual Register. The Treasurer's report shows a balance of $5,195.80 on hand December 14, 1906; of this balance $2750.06 is credited to the life-membership fund. Sales of the Society's publications during the year amounted to over $1,500.

The publication of the lectures delivered by Professors Moore, Mason, and Wilczynski at the New Haven Colloquium, September, 1906, has been generously undertaken by Yale University.

At the annual election, which closed on Saturday morning, the following officers and other members of the Council were chosen:

President,	Professor H. S. WHITE.
Vice-Presidents,	Professor HEINRICH MASCHKE,
	Professor P. F. SMITH.
Secretary,	Professor F. N. COLE.
Treasurer,	Dr. W. S. DENNETT.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor D. E. SMITH.

Members of the Council to serve until December, 1909,

Professor G. A. BLISS, Professor M. W. HASKELL,
Professor E. W. BROWN, Professor A. G. WEBSTER.

The informal dinner, always arranged in connection with each meeting of the Society, adds much to the pleasure of these occasions. Despite other distractions connected with the general gathering of scientists, over forty members attended the dinner on Friday evening and passed a few pleasant hours in social intercourse and renewal of old acquaintance.

The following papers were read at this meeting:

(1) Professor S. E. SLOCUM: "The rational basis of mathematical pedagogy."

(2) Dr. F. L. GRIFFIN: "On the law of gravitation in the binary systems."

(3) Professor JAMES MCMAHON: "A differential property of the lamellar vector field."

(4) Professor J. I. HUTCHINSON: "A method of constructing the fundamental region of a discontinuous group of linear transformations."

(5) Professor JAMES PIERPONT: "Multiple integrals" (preliminary communication).

(6) Professor OSWALD VEBLEN: "Collineations in a finite projective geometry."

(7) Dr. W. R. LONGLEY: "Some particular solutions in the problem of n bodies."

(8) Professor MAX MASON: "The expansion of an arbitrary function in terms of normal functions."

(9) Professor R. D. CARMICHAEL: "On Euler's ϕ-function."

(10) Dr. ARTHUR RANUM: "On the group of classes of congruent matrices."

(11) Dr. W. B. CARVER: "Sets of quadric spreads connected with the configuration $\Gamma^{r}_{n,r}$."

(12) Professor C. J. KEYSER: "Circle range transversals of circle ranges in a plane: a problem of construction."

(13) Professor C. J. KEYSER: "Concerning the analytic treatment of geometric involution."

(14) Dr. A. B. COBLE: "A generalization of the plane Hesse configuration."

(15) Dr. A. B. COBLE: "Involutory Cremona transformations."

(16) Professor W. E. STORY: "Denumerants of double differentiants."

(17) Professor VIRGIL SNYDER: "Birational transformations of curves of high genus."

(18) Professor T. E. McKINNEY: "On the continued fractions representing properly and improperly equivalent real numbers in a system of continued fractions depending on a variable parameter."

(19) Professor H. E. HAWKES: "On elementary divisors."

(20) Professor E. B. WILSON: "Rotations in higher dimensions."

(21) Professor EDWARD KASNER: "Systems of extremals in the calculus of variations."

(22) Professor EDWARD KASNER: "The motion of a particle in a resisting medium."

(23) Dr. R. P. STEPHENS: "Note on a system of curves of class n and order $2(n-1)$."

(24) Dr. D. C. GILLESPIE: "On the construction of an integral of Lagrange's equation in the calculus of variations."

(25) Mr. F. R. SHARPE: "The general circulation of the atmosphere."

The following papers from Section A were also placed on the programme of the Society:

Professor G. B. HALSTED: "The sect carrier and the set sect."

Professor HARRIS HANCOCK: "On a fundamental theorem of Weierstrass by means of which the theory of elliptic functions may be established."

Professor G. A. MILLER: "On the minimum number of operations whose orders exceed two in any finite group."

Dr. Gillespie's paper was communicated to the Society through Professor Snyder. Professor Slocum's paper was read by Professor Hancock. The papers of Professor Carmichael, Dr. Ranum, Dr. Coble, Professor Snyder, Professor Hawkes, Professor Wilson, Dr. Stephens, Dr. Gillespie, and Mr. Sharpe were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Slocum's article points out the lack of any general principle of mathematical pedagogy, and the desirability of establishing such a principle. The fundamental principles underlying general pedagogy are first discussed and shown to rest upon the natural law which Herbert Spencer expressed by saying that the genesis of knowledge in the individual follows the same course as the genesis of knowledge in the race. This

leads directly to the historical method of presentation which forms the basis of all scientific modern pedagogy. In the case of mathematics this method applies with peculiar force, being in fact the logical as well as the psychological method of development.

As a suggestive instance of the application of the historical method, Herbart's culture epoch theory is summarized and its merits and defects pointed out. In applying the historical method to mathematics, it is shown that the difficulties encountered in elementary branches of the subject, such as geometry and algebra, arise from a lack of harmony between the culture from which these subjects originated and the spirit of modern civilization. In this connection the culture of the Greeks, and also of the Hindoos, is characterized and contrasted with our own, thus making the nature of the difficulties encountered in these particular subjects clearly apparent. The length of time taken for the historical development of a subject is also shown to be an index of its difficulty, thus affording a means of making a quantitative, as well as a qualitative, estimate of the relative difficulty of various subjects. The various practical attempts to improve mathematical instruction, such as Grube's method of number analysis, the spiral method of instruction, the correlation of courses in the Prussian schools, the Perry movement, etc., are also studied, and shown to be but particular applications of the historical method, thus affording the best inductive proof of its validity. In conclusion, suggestions are made for a detailed application of the method to the elementary mathematical curriculum, based on the theory that the ideal arrangement of courses and subject matter depends simply on obtaining the proper historical perspective.

2. In studying the gravitational law in the binary systems, Dr. Griffin discards the more severe hypotheses hitherto made (*e. g.*, that the force is such as to make the orbit a conic for all initial conditions), and inquires what laws of force satisfy the observational data and the general hypotheses usually admitted. He shows that the only law of force under which the given orbits can be described, and satisfying the conditions: (*a*) the force is a single-valued function of the distance; (*b*) the force permits real orbits in all parts of the plane, is a type including the newtonian as a special case. Also, if the further condition be imposed, that within some circle the force be nowhere an in-

creasing function of the distance, the law of Newton is required.

A study of the trajectories for the more general type shows: that, for all initial velocities below a certain limit, the orbit is stable; and that through every point of the plane there passes in every direction one and but one orbit which is a conic. Various properties of the family of conics are pointed out; and certain exceptions, arising when the bodies are treated as having size, are discussed.

3. Professor McMahon's paper is in abstract as follows: Any lamellar field has the differential property expressed by the equation $d(\log q)/dn = d\theta/ds$, in which q is the length of the vector at a point P of the field, and $d\theta$ is the differential angle through which the vector turns as P moves through a distance ds along the curve of the vector, the other differential dn being taken along the principal normal to the curve just mentioned. A similar theorem holds in any two-dimensional lamellar field; and, in the particular case of a two-dimensional laplacian field, the differentials dn and ds may be taken in *any* two rectangular directions in the plane; this is in accordance with a well known property of two conjugate functions ($\log q$ and θ) in the complex plane. If a laplacian three-dimensional field is symmetric about an axis, it has the same differential property as a laplacian two-dimensional field if the differentiations are performed in a meridian plane.

4. The paper by Professor Hutchinson gives a simple and effective method for constructing the fundamental regions for a finite or infinite discontinuous group of linear transformations on n variables which leaves any given hermitian form unaltered.

5. Professor Pierpont's preliminary communication dealt with the relation of an improper multiple integral to its iterated integrals, in continuation of the author's paper in the *Transactions*, January, 1906.

6. Professor Veblen showed that the most general collineation of $PG(k, p^n)$ is of the form $x_i' = \sum_j a_{ij} x_j^{p^m}$, where $i, j = 1, 2, \cdots, k+1$; $m = 0, 1, \cdots, n-1$.

7. Dr. Longley considered some special cases of the plane motion of a given number of bodies. It is supposed that the

bodies form an invariable configuration which turns about the center of mass of the system. In the problem of four bodies it was shown that the rhombus is a possible configuration provided (1) opposite masses are equal, (2) the ratio of the diagonals of the rhombus is equal to or greater than unity and less than $\sqrt{3}$. The ratio of the masses is uniquely determined by the selection of the ratio of the diagonals. If a fifth body be introduced at the center of the figure, the same limits upon the ratio of the diagonals hold. A symmetric configuration was defined as one in which the masses are distributed symmetrically with respect to every line joining a body to the center of mass. A few special cases of symmetric configurations were examined. All cases in which the bodies may move in circles can be extended to motion in any conic, the configuration being always of the same shape, but of varying size.

8. In a previous paper Professor Mason gave a proof of the expansion of an arbitrary function in terms of normal functions of a certain differential equation of the second order. In that paper the convergence of the series was established by a method analogous to one used by Schmidt in connection with integral equations. In the present paper this is replaced by a simpler and more direct method.

9. Professor Carmichael's paper appeared in the February number of the BULLETIN.

10. Dr. Ranum's paper is a development of the theory begun in his article on the same subject, to appear in the *Transactions*. Among the theorems proved are the following: (*a*) Every chief binary group of classes of congruent matrices, mod $(p^{a_{ij}})(i, j = 1, 2; p = \text{prime})$, is simply isomorphic (in a certain particular way) with one and only one (normal) group, satisfying the conditions $a_{11} \geqq a_{22}$, $a_{22} \geqq a_{21} \geqq 0$, $a_{11} + a_{22} \geqq a_{12} \geqq a_{11}$. (*b*) The great majority of groups of classes of congruent matrices cannot be directly written as linear substitution groups, but every group for which the moduli of any two columns are proportional can be so written.

11. In an earlier paper,* in which the method of treatment was synthetic, Dr. Carver called attention to certain peculiarly related

* "On the Cayley-Veronese class of configurations." *Transactions Amer. Math. Society*, vol. 6, No. 4 (October, 1905).

sets of conics connected with the plane configuration $\Gamma_{n,2}^{r}$. In the present paper this idea is treated analytically and is extended to the consideration of sets of quadric spreads connected with the r-dimensional configuration $\Gamma_{n,r}^{r}$. It is shown that, the quadric spreads being given, the configuration is determined. Special cases for $r = 1$ and $r = 3$ are then discussed.

12. Professor Keyser's first paper deals with an important detail in the plane geometry that arises from employing the circle range as primary element. A circle range is the ensemble of circles having two points in common. Two circle ranges having a common circle are transversals of each other. Four arbitrary circle ranges r_1, r_2, r_3, r_4 have in common two and but two transversal ranges, ρ_1 and ρ_2. These may be real and distinct or coincident or conjugate imaginary. The problem solved is that of constructing ρ_1 and ρ_2 in the case where they are real. The four ranges taken in sets of three determine four semiquadrics of ranges, viz., Q_{123}, Q_{124}, Q_{134}, Q_{234}. The transversals of the ranges of a Q constitute a complementary semiquadric Q' of that Q. The four Q's have ρ_1 and ρ_2 in common. The four Q's have no common range, but any three of them have one, any two of them two, common ranges. Any Q and its Q' together constitute a hyperboloidal configuration H of ranges, having thus two sets of generating ranges, those of Q and those of Q'. The four H's have ρ_1 and ρ_2, but no other ranges, in common. The radical axes of the ranges of a Q envelop a conic section. The like is true of a Q'. If a Q and a Q' belong to the same H, the two conics coincide. The four conics thus associated with the four H's have the radical axes of ρ_1 and ρ_2 in common. The conics are the intersections of the H's and the special congruence composed of the lines (infinite circles) of the plane. The conic as envelope is thus a second order locus of infinite circles. In like manner the point conic is the envelope of circle ranges of infinite circles. The four conics may be constructed by using the radical axes of ρ_1 and ρ_2 and those of the given r's taken three at a time. The problem of constructing ρ_1 and ρ_2 is *logically* the same as that of constructing the transversals of four lines of space, but the construction makes no use in euclidean fashion of any circle configuration analogous to the plane, a fact supporting the hypothesis that the plane need not be employed as element in the three-dimensional constructions.

13. The second paper by Professor Keyser deals with the question of a suitable algebra for geometric involution. Connecting the exposition (for the sake of convenience) with a range of points, the problem is to provide an analytic machinery that shall handle the point pair as element instead of the points of the pair. This is done in a manner analogous to that of Darboux and Klein in the matter of pentaspherical point and sphere coordinates in space. Three point pairs in mutual involution are taken for configuration of reference. The power of a point as to a pair is the product of the distances between the points and the points of the pair. If π_1, π_2, π_3 be the powers of P as to the fundamental pairs, and if α_1, α_2, α_3 be the reciprocals of the half distances of the points of the fundamental pairs, then $\sum(\alpha_i\pi_i)^2 = 0$. If (1) $\lambda x = \alpha_i\pi_i$, then (2) $\sum x_i^2 = 0$. Equations (1) and (2) suffice to determine the x_i as a redundant system of point coordinates. An equation of the form $\sum c_i x_i = 0$ represents a point pair and conversely. Two point pairs ξ_i and η_i are in involution when and only when (3) $\sum\xi_i\eta_i = 0$. If the ξ_i be fixed and the η_i variable, (3) defines an involution ξ_i. The ξ_i are also the coördinates of the focal pair. If ξ_i be variable and η_i fixed, (3) defines a pencil η_i of involutions. The η_i are also the coordinates of the two focal involutions of the pencil. Thus is evidenced the bidimensionality of the range in involutions and in pencils (of involutions) and the dual relationship of these entities as elements. The geometry thus arising in the line (or other like space) is quite as rich as and is algebraically identical with the range-pencil geometry of the plane or any similar doctrine. Ordinarily involution appears as a mere detail of projective geometry. That is because, as ordinarily presented, it is but the intersection of the theory here indicated and the usual projective geometry of the plane. In the geometry here in question, every involution contains one infinite pair (composed of a finite point and the infinite point of the line) and two zero-radial pairs (points of pair falling together). If the ensemble of infinite pairs be taken for absolute, the geometry will be euclidean or parabolic. It will be lobachevskian or hyperbolic if the ensemble of zero-radial pairs be taken for absolute.

14. In Dr. Coble's first paper it is noted that two interchangeable plane collineations of period 3, which are not powers of each other, and which have unique fixed triangles, uniquely

define a plane Hesse configuration. An immediate generalization can be made in spaces of $p-1$ dimensions S_{p-1}, p being any prime number greater than 2. The resulting configuration is made up of p^2 $S_{\frac{1}{2}(p-3)}$'s and p^2 $S_{\frac{1}{2}(p-1)}$'s so situated that the $S_{\frac{1}{2}(p-3)}$'s lie, in sets of p, in $p(p+1)$ S_{p-2}'s; and the $S_{\frac{1}{2}(p-1)}$'s meet, in sets of p, at $p(p+1)$ S_0's. The collineation group associated with the configuration has an invariant subgroup of order $2p^2$, and the factor group is isomorphic with the linear group $LF(2, p)$.

15. Involutory Cremona transformations have been divided by Bertini into four classes according as they are reducible by a Cremona transformation to one or another of four "types." The object of Dr. Coble's second paper is to study the general case of a class by using a construction of the involution which is unaltered by Cremona transformation. In the cases considered the dependence of the fundamental points upon each other is given. The invariant curves and allied line loci are treated in some detail.

16. An algebraic form (homogeneous and isobaric polynomial) in any number of variables x, $y, \cdots$ and in the coefficients of any number of quantics of any orders that is not altered by such a linear transformation as simply adds an arbitrary constant multiple of x to y may be called an xy-differentiant belonging to the system of quantics, and the property just described may be called the differentiant character xy. In 1854 Cayley published a formula for the number of linearly independent xy-differentiants of any given type that belong to a single binary quantic of any order, and this formula was proved and extended to any number of binary quantics by Sylvester in 1877. The object of Professor Story's paper is to extend Cayley's formula to forms in any number of variables and in the coefficients of any number of quantics of any orders that have any *two* given differentiant characters. The denumerants actually given are those for xy- and xz-differentiants, for xz- and yz-differentiants, for xy- and yz-differentiants (which are also xz-differentiants), and for xy- and zy-differentiants, without regard to the whole number of variables involved. The denumerant for in- and covariants of any system of ternary quantics is a particular case of that for xy- and yz-differentiants. Incidentally, a number of properties of differentiants are estab-

lished. The general method of determining denumerants here developed is applicable to any number of differentiant characters, and the writer hopes before long to extend his formulas to triple differentiants, including in- and covariants of quaternary systems.

17. By means of plane sections of ruled surfaces, and of special quadric spreads in hyperspace, Professor Snyder proves that when the points of two nonsingular curves are in (1, 1) correspondence they are projectively equivalent. More generally, if a curve c_n has an i-fold point ($i \leqq n-3$) and no other singularity, it cannot be transformed birationally into itself or any other curve of order n except by collineation; similarly, if it have a k-fold point ($k \leqq n-4$) and a double point, or not more than $E[(\frac{1}{2}(n-1))^2]-2$ distinct double points, $E(k)$ being the largest integer in k.

18. In Professor McKinney's paper the axis of reals is divided into intervals of the types $a)$ $a_i - 1 + \lambda \cdots (a_i + \lambda)$, $a_i > 0$; $b)$ $(a_i - \lambda) \cdots a_i + 1 - \lambda$, $a_i < 0$; $c)$ $(a_i - \lambda) \cdots (a_i + \lambda)$, $a_i = 0$, where a_i is an integer, λ a variable parameter, $0 < \lambda \leqq 1$, and the effect of the parentheses is to exclude the inclosed number from the interval. The system of intervals is symmetric with respect to the origin. Every number in the interval is said to correspond to a_i, the representative number of the interval. The defining equation takes the form $x_i = a_i - 1/x_{i+1}$, where x_i, real and not zero, corresponds to a_i.

For all values of λ, $\frac{1}{2}(-1+\sqrt{5}) < \lambda \leqq 1$, the continued fractions representing two properly equivalent numbers are ultimately alike. For $0 < \lambda \leqq \frac{1}{2}(-1+\sqrt{5})$ they are ultimately alike except under specified conditions. In the case excepted the continued fraction representing one of the numbers may, from elements of a certain rank on, be readily derived from that representing the other number. The results for improper equivalence follow immediately from the relation of the continued fractions representing x and $-x$.

19. Professor Hawkes's paper gave a proof of the necessary and sufficient conditions for the equivalent of two families of bilinear forms when the determinant of the family is not zero. The proof depends on the reduction of a matrix to normal forms, in

which appear numbers that are precisely the exponents of the elementary divisors of the characteristic determinant of the matrix.

20. Professor Wilson places the theory of rotations in connection with the general theory of matrices or dyadics from the point of view of Gibbs. This leads to an easy and immediate treatment of such questions as those treated in the case of four dimensions by F. N. Cole, *American Journal of Mathematics*, volume 12 (1890). The treatment given by Professor Wilson leads to the general classification and reduction of n-dimensional pairs in case the matter of reality is regarded as important.

21. Professor Kasner's first paper appears in the present number of the BULLETIN.

22. In recent papers published in the *Transactions* Professor Kasner has investigated the characteristic geometric properties of the system of trajectories connected with any positional field of force. In the present paper the particle is acted upon by such a force and also by a resisting medium. Distinct discussions are necessary for the two-dimensional and three-dimensional problems. In both discussions the case where the resistance varies as the square of the velocity is especially interesting since then and only then it turns out that certain of the properties which hold for free motion hold also in the resisted motion. These are the properties numbered I and II in the papers referred to above.

23. Dr. Stephens showed that, if a point t of a circle be projected at a fixed inclination upon the n Wallace lines which arise from n points t_i of the same circle taken $n-1$ at a time, then these n projections lie in a straight line whose envelope, as t varies, is a curve of class n and order $2(n-1)$. If the angle of projection be allowed to vary, there is formed a system of curves whose centers lie on a line. If $n=3$, Dr. Converse's case results at once; if the points t_i form a regular polygon, we have the case considered by Professor Steggall.

24. If the vanishing of the first variation of two known integrals each containing one dependent variable gives the same differential equation, then by a theorem of Darboux it is possible to write immediately an intermediary integral of this

differential equation. The paper of Dr. Gillespie shows that if two known integrals, each containing n dependent variables, give through the vanishing of their first variation the same system of differential equations, it is then possible to write immediately an integral of this system of differential equations.

25. The general circulation of the atmosphere has been discussed by Ferrel and Oberbeck. The former made no attempt at a complete solution. The latter assumed that the air was incompressible but satisfied Boyle's law. He also surrounded the atmosphere with a spherical boundary of indefinite height. In Mr. Sharpe's solution it is shown that when there is no inequality of temperature between the equator and the poles the air is in both adiabatic and conductive equilibrium. Consequently when there is an increase of temperature from the pole to the equator the temperature still satisfies Fourier's law of conduction and a circulation exists which vanishes at the upper limit of the atmosphere as well as on the earth's surface. This meridional circulation modifies the east and west motion of the air, which in turn changes the pressure distribution and the circulation.

F. N. COLE,
Secretary.

THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE twentieth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, Chicago, Ill., on Friday, December 28, 1906. The total attendance was thirty-five, including the following twenty-nine members of the Society:

Mr. G. D. Birkhoff, Professor G. A. Bliss, Professor Oskar Bolza, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Otto Dunkel, Professor T. F. Holgate, Mr. Louis Ingold, Professor O. D. Kellogg, Professor Kurt Laves, Mr. N. J. Lennes, Dr. A. C. Lunn, Mr. W. S. MacMillan, Professor H. Maschke, Professor G. A. Miller, Professor E. H. Moore, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor H. L. Rietz, Professor N. C. Riggs, Mr. W. J. Risley, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. C. H.

Sisam, Professor H. E. Slaught, Mr. R. E. Wilson, Professor J. W. A. Young, Professor Alexander Ziwet.

Two sessions were held for the reading of papers, at both of which the chairman of the Section, Professor Alexander Ziwet, presided. In the evening seventeen members of the Section dined together and enjoyed a season of social intercourse.

The committee having in charge the arrangement of a meeting for the discussion of mathematics for engineers reported that the proposed plans had been modified on account of the conflict in time with the meeting of the American association for the advancement of science in New York. In view of the announcement that this association is to meet in Chicago in December, 1907, the committee was instructed to arrange if possible a joint session of the Chicago Section with Sections A and D of the American association and with the Society for the promotion of engineering education, which meets at the same time and place.

Professor E. B. Van Vleck was elected Chairman and Professor H. E. Slaught Secretary of the Section, for the ensuing year. These, together with Professor H. L. Rietz, constitute the programme committee; and, with the further addition of Professors A. G. Hall, E. J. Townsend, E. B. Skinner and Alexander Ziwet, constitute the committee on programme for the December, 1907, meeting.

The following papers were read:

(1) Professor JAMES B. SHAW: "On the equivalence of algebras."

(2) Professor JAMES B. SHAW: "Quaternion associative lineolinear expressions."

(3) Professor O. BOLZA: "On Lagrange's method of multipliers in the calculus of variations."

(4) Dr. C. H. SISAM: "On a locus associated with plane curves."

(5) Professor G. A. MILLER: "The groups generated by two operators such that each is transformed into its inverse by the square of the other."

(6) Mr. A. R. SCHWEITZER: "On an interesting class of monotonic functions."

(7) Professor L. E. DICKSON: "Invariantive theory of quadratic forms in a finite field."

(8) Professor D. R. CURTISS: "Sufficient conditions for the linear dependence of functions of more than one variable."

(9) Professor G. A. BLISS: "Note on maxima and minima of functions of several variables."

(10) Professor O. D. KELLOGG: "The Green formula

$$u(x, y) = \frac{1}{2\pi}\int_C u(\sigma)\,\frac{\partial}{\partial \nu} G(x, y;\ \xi, \eta)d\sigma."$$

(11) Dr. A. C. LUNN: "An example of a mechanical system with numerably infinite degrees of freedom."

Abstracts of the papers follow below, the numbers corresponding to the titles in the list above.

1. This first paper of Professor Shaw defines the equivalence of two algebras, thus: Two algebras are equivalent when, if they are supposed to have the same fundamental units, and if the product of two numbers α and β is indicated for one algebra by $\Phi\ \alpha\beta$, for the other by $\Phi' \cdot \alpha\beta$, then

$$\Phi' \cdot \alpha\beta = \phi\Phi\phi^{-1}\alpha\phi^{-1}\beta,$$

where ϕ is any linear operator which transforms the units $e_1, \cdots, e_r$ into numbers ϵ given by the equations

$$e_i = \phi\epsilon_i = \sum_{j=1}^{r} c_{ij}\epsilon_j \qquad (i = 1, \cdots, r).$$

It is understood that $|\phi| = |c_{ij}| \neq 0$, that is, ϕ has no zero roots.

The consequences of this definition are developed, with some consideration of the two problems: (1) to simplify the defining equations of an algebra by removing unessential parameters; (2) to determine algebras whose form is in some degree invariant under the transformation ϕ, these algebras furnishing means of defining other algebras without reference to a multiplication table.

2. The second paper of Professor Shaw develops all lineo-linear functions of two quaternions q, r which are associative, that is, for which

$$F(q, F[r, s]) = F(F[q, r], s).$$

3. Professor Bolza's paper gives a proof of Lagrange's

rule of multipliers and of the limit equations for the problem of minimizing the integral

$$I = \int_{t_0}^{t_1} f(y_1, y_2, \cdots, y_n ; y_1', y_2', \cdots, y_n')dt$$

when the admissible curves are subject to the condition of satisfying p differential equations

$$\phi_\alpha(y_1, y_2, \cdots, y_n ; y_1', y_2', \cdots, y_n') = 0 \quad (\alpha = 1, 2, \cdots, p)$$

and q finite equations

$$\psi_\beta(y_1, y_2, \cdots, y_n) = 0 \quad (\beta = 1, 2, \cdots, q),$$

while the coördinates of their end points $(y_{10}, y_{20}, \cdots, y_{n0})$, $(y_{11}, y_{21}, \cdots, y_{n1})$ satisfy r relations of the form

$$\chi_\gamma(y_{10}, y_{20}, \cdots, y_{n0}, y_{11}, y_{21}, \cdots, y_{n1}) = 0 \quad (\gamma = 1, 2, \cdots, r).$$

The proof is based upon an extension of the method used by Hilbert for the special case when $q = 0$ and the end points are fixed. The paper will appear in the *Mathematische Annalen*.

4. Dr. Sisam's paper treats of the locus of the points of intersection of the tangents to an arbitrary algebraic curve whose points of tangency lie on a common tangent to a second algebraic curve. The order, and other interesting properties of this locus, and the corresponding properties of the dual configuration are derived. Two particular cases, first when the second of the given curves reduces to a point, and second when the locus curve coincides with the first of the two given curves, are studied in detail.

5. The main theorems proved in Professor Miller's paper may be stated as follows: If two operators neither of which is identity are such that each is transformed into its inverse by the square of the other, they must have the same order and this common order is 2, 4 or 8. If the common order exceeds two, the group generated by them involves operators of order 3; in fact, this group must have an $(a, 1)$ isomorphism with the symmetric group of order 24. The smallest possible group which is generated by two such operators of order 8 is the group of order 48 known as G_{52}, and every other group that

can be generated by two such operators has an $(\alpha, 1)$ isomorphism with this group of order 48. The paper also contains a new and useful definition of the symmetric group of order 24. This paper is to be offered to the *Annals of Mathematics* for publication.

6. Representing the points $0 \leqq x \leqq 1$ by the series

$$x = \sum_{k=1}^{+\infty} \frac{\alpha_k}{m_1 m_2 \cdots m_k},$$

where the numbers α_k, m_k are integers, $0 \leqq \alpha_k \leqq m_k - 1$, $m_k \geqq 2$, Mr. Schweitzer discusses the function

$$t(x) = \sum_{k=1}^{+\infty} r_k \frac{\alpha_k^{l_k}}{m_1' m_2' \cdots m_k'},$$

where l_k, m_k' are integers. The paper will be offered to the *Annals of Mathematics* for publication.

7. In the paper by Professor Dickson there is established a theorem on m-ary quadratic forms in the $GF[2^n]$ analogous to that concerning the rank r of the discriminantal determinant Δ in the algebraic theory and the minimum number r of variables on which the form can be expressed. In place of Δ (for m odd) and of all minors of odd order, we employ the halves of their algebraic expansions, necessarily with even coefficients. In particular, the semi-discriminant is an invariant. When the rank r is even, there are two classes of forms, distinguishable by an arithmetic property of $\Delta^{2^{n-1}}$. Following this preliminary part, the paper proceeds to the determination and interpretation of a complete set of linearly independent invariants of the quadratic form. This invariantive theory differs in essential features from the simple algebraic theory. The paper will be offered for publication to the *American Journal of Mathematics.*

8. If $u_1, u_2, \cdots, u_m$ are m functions of n variables all of whose partial derivatives of the first $m - 1$ orders exist and are finite throughout a given region, a necessary condition for their linear dependence is the vanishing of all m-rowed determinants of the matrix whose first row consists of the m functions u, and whose succeeding rows are composed of partial derivatives of the elements of the first row, so that to each deriv-

ative of the first $m-1$ orders there corresponds a row. In Professor Curtiss's paper some properties of matrices of this sort are developed and sufficient conditions for the linear dependence of the functions u are given.

9. In investigating whether or not a function $f(x_1, x_2, \cdots, x_n)$ has a maximum or minimum at a point $(a_1, a_2, \cdots, a_n)$, it is usual to suppose that all the derivatives of the function of order less than n vanish at the point $(a_1, a_2, \cdots, a_n)$, while one at least of order n is different from zero. Then the function is expanded by Taylor's formula with a remainder to terms of order $n+1$, or else in an infinite series. In Professor Bliss's paper it is shown how the usual results in the definite and indefinite cases can be obtained from Taylor's expansion to terms of order n only. The use of derivatives of higher orders than those occurring in the criteria which enable one to decide if there is a maximum or minimum is thus avoided. This paper will be offered to the BULLETIN for publication.

10. If $u(x, y)$ be a potential function of a simply connected closed region R of the (x, y)-plane bounded by a regular closed curve C, then it is known that if u and its first derivatives are continuous on the boundary of R, it may be represented in the form

$$u(x, y) = \frac{1}{2\pi}\int_C u(\sigma)\frac{\partial}{\partial \nu} G(x, y\,;\, \xi, \eta)d\sigma,$$

exhibiting the dependence of the function upon its boundary values $u(\sigma)$. Starting on the other hand from this formula, the question arises: under what conditions upon boundary curve C and upon boundary values $u(\sigma)$ is the above integral a potential function of R which approaches the boundary values $u(\sigma)$? For regions bounded by curves the coordinates of whose points as functions $x(s)$, $y(s)$ of the length of arc have uniformly "algebraically continuous" derivatives (*i. e.*, $|\Delta x'(s)| < A|\Delta s|^\alpha$, $|\Delta y'(s)| < A|\Delta s|^\alpha$, $0 < \alpha$ so soon as $|\Delta s| < \delta$), Professor Kellogg shows that a potential function is determined by a given set of boundary values through the above formula under the same conditions upon $u(\sigma)$ as in the case of the circle through Poisson's integral.

11. Dr. Lunn's paper gives the solution of an infinite system of ordinary differential equations, which are shown to determine

the motion of an infinite linear system of discrete masses, connected by springs. The solution is obtained indirectly by a limiting process from the solution for a finite number of masses, and is then verified directly. The main features of the oscillations of a given mass are interpreted in terms of familiar properties of the Bessel functions of the time which occur as coefficients.

H. E. SLAUGHT,
Secretary of the Section.

THE DECOMPOSITION OF MODULAR SYSTEMS CONNECTED WITH THE DOUBLY GENERALIZED FERMAT THEOREM.

BY PROFESSOR ELIAKIM HASTINGS MOORE.

(Read before the Chicago Section of the American Mathematical Society, December 29, 1898.)

Introduction. The Generalized Fermat Theorem (A) *in Purely Arithmetic Phrasing* (A', A'') *with Extension* (A'''). §§ 1–5.

1. The theorem * in question is the following:

(A) In the Galois field $GF[p^n]$ of prime modulus p and of rank n the two forms each of degree $(p^{n(k+1)}-1)/(p^n-1)$ in the $k+1$ indeterminates $X_0, X_1, \cdots, X_k$

$$D_{k+1,n,p}[X_0, X_1, \cdots, X_k] = |X_j^{p^{in}}| \qquad (i, j = 0, 1, \cdots, k)$$

$$P_{k+1,n,p}[X_0, X_1, \cdots, X_k] = \prod_{g=0,k} \prod_{\alpha_{fg}|p^n} (X_g + \sum_{f=0,g-1} \alpha_{fg} X_f)$$

are identical:

$$D_{k+1,n,p}[X_0, \cdots, X_k] = P_{k+1,n,p}[X_0, \cdots, X_k].$$

Here the subscript remark $\alpha_{fg}|p^n$ indicates that the mark α_{fg} is to run over the p^n marks of the Galois field $GF[p^n]$, and for the case $g=0$ the final $\sum_{f=0,g-1}$ does not enter.

For this theorem, which for $(k, n) = (1, 1)$ is one form of Fermat's theorem, I have given three proofs, couched as is the statement of the theorem in the abstract Galois field phrasing introduced by me in the paper "A doubly-infinite system of simple groups" presented to the Chicago Congress of 1893.

* Moore, "A two-fold generalization of Fermat's theorem," BULLETIN, vol. 2 (1896), pp. 189–199.

In the development of the Galois field theory and in its applications to algebra and groups this abstract phrasing is very convenient.

2. Here however we are interested in converses of the theorem stated above; after replacing the marks α_{fg} by functions of the realm $[1, y]$ we decompose, in the sense of equivalence, into prime modular systems the modular system $\mathfrak{M}$ whose elements are the coefficients of various powers of the indeterminates in the form

$$M_{k+1,\,n,\,p} \equiv D_{k+1,\,n,\,p} - P_{k+1,\,n,\,p}.$$

We need first to use the concrete purely arithmetic phrasings of Serret and Kronecker. The $GF[p^n]$ is then the totality of p^n classes (abstractly, marks) of rational integral functions of an indeterminate y with integral coefficients (forming the realm of integrity $[1, y]$) considered with respect to a (prime) modular system $[p, F_n[y]]$, where $F_n[y]$ is a function of the realm $[1, y]$ of degree n (a polynomial in y with integral coefficients) irreducible modulo p. We operate with these classes by operating with representative functions with respect to the modular system $[p, F_n[y]]$. For the marks α we may take the p^n reduced functions

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{n-1} y^{n-1},$$

where the n coefficients $a_0, \cdots, a_{n-1}$ take independently the values $0, 1, 2, \cdots, p-1$. There are in this sense as many concretely distinct Galois fields as there are functions $F_n[y]$ congruentially distinct modulo p. These Galois fields are however abstractly identical.

We restate theorem

(A') In the realm of integrity $[1, y]$ the two forms each of degree $(p^{n(k+1)} - 1)/(p^n - 1)$ in the $k+1$ indeterminates $X_0, \cdots, X_k$

$$D_{k+1,\,n,\,p}[X_0, \cdots, X_k] = |X_j^{p^{ni}}| \quad (i, j = 0, 1, \cdots, k)$$

$$P_{k+1,\,n,\,p}[X_0, \cdots, X_k] = \prod_{g=0,\,k} \prod_{a_{fgl}=0,\,p-1} \left(X_g + \sum_{f=0,\,g-1} X_f \sum_{l=0,\,n-1} a_{fgl} y^l\right)$$

are identically * congruent ($\equiv$):

* In pure arithmetic all identities and identical congruences are *formal* in certain specified indeterminates.

$$D_{k+1,n,p}[X_0, \cdots, X_k] \equiv P_{k+1,n,p}[X_0, \cdots, X_k] \quad [p, F_n[y]],$$

with respect to every prime modular system $[p, F_n[y]]$, where p is a prime and $F_n[y]$ is any function of $[1, y]$ of degree n and irreducible modulo p.

3. I take as known the fundamental definitions and elementary theorems of Kronecker's theory of modular systems, in particular with respect to the composition and equivalence of modular systems. It is however desirable to fix the notations * to be used here.

A realm $\mathfrak{R}$ of integrity-rationality

$$\mathfrak{R} = [R_1 \cdots, R_\mu](R_{\mu+1}, \cdots, R_{\mu+\nu})$$

consists of all functions

$$F[R_1, \cdots, R_\mu](R_{\mu+1}, \cdots, R_{\mu+\nu})$$

rational integral in $R_1, \cdots, R_\mu$ and rational in $R_{\mu+1}, \cdots, R_{\mu+\nu}$, the coefficients being integers. These functions are called the *quantities* of the realm. The realm is closed under addition, subtraction, and multiplication, and likewise under division by any function not 0 of $\mathfrak{R}' = (R_{\mu+1}, \cdots, R_{\mu+\nu})$.

Any set of quantities $F_1, \cdots, F_m$ of a realm $\mathfrak{R}$ constitutes a modular system $\mathfrak{F} = [F_1, \cdots, F_m]$ of that realm. The whole theory of such modular systems relates to the underlying realm.

Any set of modular systems

$$\mathfrak{F}_i = [F_{i1}, \cdots, F_{im_i}] \qquad (i = 1, 2, \cdots, m)$$

determines a modular system $[\cdots F_{ij_i} \cdots {}^{i=1,\cdots,m}_{j_i=1,\cdots,m_i}]$ for which we use the notation $[\mathfrak{F}_1, \cdots, \mathfrak{F}_m]$.

(1) The theorem:

If

$$[\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}] \sim [1],$$

then

$$[\mathfrak{F}_1, \mathfrak{F}][\mathfrak{F}_2, \mathfrak{F}] \sim [\mathfrak{F}_1\mathfrak{F}_2, \mathfrak{F}],$$

and its useful generalization:

If

$$[\mathfrak{F}_i, \mathfrak{F}_j, \mathfrak{F}] \sim [1] \qquad (i \neq j;\ i, j = 1, \cdots, m),$$

* The notations are those of my paper "The decomposition of modular systems of rank n in n variables," BULLETIN, vol. 3 (1897), pp. 372-380.

then

$$\prod_{i=1,\,m} [\mathfrak{F}_i, \mathfrak{F}] \sim [\prod_{i=1,\,m} \mathfrak{F}_i, \mathfrak{F}]:$$

may readily be proved.

(2) A system $\mathfrak{F}$ of the realm $\mathfrak{R}$ I call *prime** in case (α) $\mathfrak{F}$ and [1] are not equivalent and (β) for any quantity G of the realm we have one of the alternative equivalences

$$[\mathfrak{F}, G] \sim \mathfrak{F} \quad \text{or} \quad [\mathfrak{F}, G] \sim [1],$$

that is, one of the alternative congruences

$$G \equiv 0 [\mathfrak{F}] \quad \text{or} \quad GG' \equiv 1 [\mathfrak{F}],$$

where in the latter case G' is a properly determined quantity of the realm.

For a prime system $\mathfrak{F}$ in connection with any system $\mathfrak{G}$ we have one of the alternatives

$$[\mathfrak{F}, \mathfrak{G}] \sim \mathfrak{F} \quad \text{and} \quad \mathfrak{G} \equiv 0 [\mathfrak{F}] \quad \text{or} \quad [\mathfrak{F}, \mathfrak{G}] \sim [1],$$

that is, any system $\mathfrak{G}$ contains or is relatively prime to a prime system $\mathfrak{F}$.

We have the fundamental theorem:

If $\mathfrak{F}$ is prime and the product $\mathfrak{G}_1\mathfrak{G}_2$ of two systems $\mathfrak{G}_1, \mathfrak{G}_2$ contains $\mathfrak{F}$, while $\mathfrak{G}_1$ does not contain $\mathfrak{F}$, then $\mathfrak{G}_2$ does contain $\mathfrak{F}$.

For $\mathfrak{G}_1\mathfrak{G}_2 \equiv 0 [\mathfrak{F}]$ and $\mathfrak{G}_1 \not\equiv 0 [\mathfrak{F}]$ and hence, since $\mathfrak{F}$ is prime, $[\mathfrak{F}, \mathfrak{G}_1] \sim [1]$. Now $\mathfrak{F}\mathfrak{G}_2 \equiv 0 [\mathfrak{F}]$. Hence we have $[\mathfrak{F}, \mathfrak{G}_1]\mathfrak{G}_2 \equiv 0 [\mathfrak{F}]$, and so indeed $\mathfrak{G}_2 \equiv 0 [\mathfrak{F}]$.

(3) If $[\mathfrak{F}, \mathfrak{G}_i] \sim [1]$ $(i = 1, 2, \cdots, m)$, then

$$[\mathfrak{F}, \prod_{i=1,\,m} \mathfrak{G}_i] \sim \prod_{i=1,\,m} [\mathfrak{F}, \mathfrak{G}_i] \sim [1].$$

(4) If $[\mathfrak{F}_i, \mathfrak{F}_j] \sim [1]$ $(i \neq j;\ i, j = 1, 2, \cdots, m)$ and $\mathfrak{M} \equiv 0$ $[\mathfrak{F}_i]$ $(i = 1, 2, \cdots, m)$, then $\mathfrak{M} \equiv 0$ $[\mathfrak{F}]$ where $\mathfrak{F} = \prod_i \mathfrak{F}_i$.

4. We are now to work in the realm of integrity $\mathfrak{R}_1 = [1, y]$, where y is an indeterminate. There are in all say m functions $F_n[y]$ of $\mathfrak{R}_1$ congruentially distinct (mod. p) and each of degree n and irreducible mod. p; we write them $F_{nj}[y] (j = 1, \cdots, m)$. The modular systems $\mathfrak{F}_{nj} = [p, F_{nj}[y]] (j = 1, \cdots, m)$ are

* Kronecker's prime modular system of a given rank (*Jour. für Mathematik*, vol. 99, p. 337) is differently defined.

all prime and by pairs relatively prime. We write their product

$$\mathfrak{F}_n = \prod_{j=1,\,m} \mathfrak{F}_{nj} = \prod_{j=1,\,m} [p, F_{nj}[y]] \sim [p, \prod_{j=1,\,m} F_{nj}[y]].$$

5. We set (using the notations of § 2)

$$M_{k+1,n,p}[X_0, \cdots, X_k] = D_{k+1,n,p}[X_0, \cdots, X_k] - P_{k+1,n,p}[X_0, \cdots, X_k]$$

and denote by $\mathfrak{M}_{k+1,n,p}$ the modular system of coefficients of $M_{k+1,n,p}[X_0, \cdots, X_k]$; this system $\mathfrak{M}$ belongs to the realm $\mathfrak{R}_1 = [1, y]$.

Our theorem A'(§ 2) is then

$$(A'') \qquad \mathfrak{R}_{k+1,n,p} \equiv 0 \ [\mathfrak{F}_{nj}] \qquad (j = 1, \cdots, m),$$

whence follows in accordance with § 4 and § 3 (4),

$$(A''') \qquad \mathfrak{M}_{k+1,n,p} \equiv 0 \ [\mathfrak{F}_n].$$

The Equivalence Theorem (*B*) *and the Decomposition Theorem* (*C*) §§ 6–14.

6. We are to prove the equivalence

$$(B) \qquad \mathfrak{M}_{k+1,n,p} \sim \mathfrak{F}_n,$$

whence in view of § 4 follows the decomposition, in the sense of equivalence, of the system $\mathfrak{M}_{k+1,n,p}$, viz.,

$$(C) \qquad \mathfrak{M}_{k+1,n,p} \sim \prod_{j=1,\,m} \mathfrak{F}_{nj},$$

that is, its exhibition as a product of systems no further decomposable and indeed in this case prime.

7. To prove (*B*) we need to prove, (A''') being admitted, merely the converse of (A'''), viz.,

$$(\bar{A}''') \qquad \mathfrak{F}_n \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}].$$

Here $\mathfrak{F}_n$ depends upon n and p and not on k. The congruence ($\bar{A}'''$) follows from the following two congruences:

$$(\bar{A}'''_{k=1}) \qquad \mathfrak{F}_n \equiv 0 \quad [\mathfrak{M}_{2,n,p}],$$

$$(D) \qquad \mathfrak{M}_{2,n,p} \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}],$$

of which the former is the particular case of $(\bar{A}''')$ for $k = 1$. We shall, however, prove (D) and then (B, C) without the mediation of $(\bar{A}'''_{k=1})$.

8. Proof of (D). The obviously true congruence

$$\mathfrak{M}_{k+1,\,n,\,p} \equiv 0 \quad [\mathfrak{M}_{k+1,\,n,\,p}]$$

implies the identical congruence

$$M_{k+1,\,n,\,p}[X_1, X_1, \cdots, X_0] \equiv 0 \quad [\mathfrak{M}_{k+1,\,n,\,p}]$$

in the $k+1$ indeterminates $X_0, \cdots, X_k$. Considering in particular the terms of M with the literal part containing the factor $X_2^{p^{2n}} \cdots X_i^{p^{in}} \cdots X_k^{p^{kn}}$, we have the identical congruence in X_0, X_1

$$(X_0 X_1^{p^n} - X_1 X_0^{p^n}) - \prod_{\substack{a_0 \cdots a_{n-1} \\ =0,\,p-1}} (X_1 + X_0 a_0 + a_1 y + \cdots + a y^i + \cdots + a_{n-1} y^{n-1}) \equiv 0 \quad [\mathfrak{M}_{k+1,\,n,\,p}],$$

that is,

$$M_{2,\,n,\,p}[X_0, X_1] \equiv 0 \quad [\mathfrak{M}_{k+1,\,n,\,p}],$$

and so in fact we have

$$(D) \qquad \mathfrak{M}_{2,\,n,\,p} \equiv 0 \quad [\mathfrak{M}_{k+1,\,n,\,p}].$$

9. We consider in preparation for the proof of (B, C) (§ 11) various properties of the system $\mathfrak{M} = \mathfrak{M}_{k+1,\,n,\,p}$, consequences of the identical congruence

$$M[X_0, X_1] = D[X_0, X_1] - P[X_0, X_1] = (X_0 X_1^{p^n} - X_1 X_0^{p^n}) - \prod_{\substack{a_0 \cdots a_{n-1} \\ =0,\,p-1}} (X_1 + X_0 a_0 + a_1 y + \cdots + a_i y^i + \cdots + a_{n-1} y^{n-1}) \equiv 0 \,[\mathfrak{M}];$$

this congruence obviously true for $\mathfrak{M} = \mathfrak{M}_{2,\,n,\,p}$ is by (D) true for $\mathfrak{M} = \mathfrak{M}_{k+1,\,n,\,p}$.

(1) G being any quantity $G = a_0 + a_1 y + \cdots + a_{n-1} y^{n-1}$ of our realm $\mathfrak{R}_1 = [1, y]$ of degree in y at most $n-1$ with coefficients a_i each 0 or positive integers less than p but not all 0, we have

$$[\mathfrak{M}, G] \sim [1],$$

and so there is a quantity G' of $\mathfrak{R}_1$ such that

$$GG' \equiv 1 \quad [\mathfrak{M}].$$

Proof. Since $\mathfrak{M} \equiv 0 \quad [\mathfrak{M}, G]$ we have

$$D[X_0, X_1] \equiv P[X_0, X_1] \quad [\mathfrak{M}, G].$$

Now $P[X_0, X_1]$ has the distinct factors X_1 and $X_1 + GX_0$ and other factors with product say $Q[X_0, X_1]$; hence

$$P[X_0, X_1] \equiv X_1^2 Q[X_0, X_1] \quad [G]$$

and so, since $G \equiv 0 [\mathfrak{M}, G]$,

$$P[X_0, X_1] \equiv X_1^2 Q[X_0, X_1] \quad [\mathfrak{M}, G].$$

Hence

$$D[X_0, X_1] \equiv X_1^2 Q[X_0, X_1] \quad [\mathfrak{M}, G].$$

The term $X_1 X_0^{p^n}$ gives the congruence

$$-1 \equiv 0 \quad [\mathfrak{M}, G],$$

that is, $[1] \equiv 0 [\mathfrak{M}, G]$, and, since $[\mathfrak{M}, G] \equiv 0 [1]$, we have proved that $(\mathfrak{M}, G) \sim [1]$.

(2°) There is an integer g for which

$$pg \equiv 0 \quad [\mathfrak{M}] \quad \text{and} \quad [p, g] \sim [1].$$

Proof. From the identical congruence by the substitution $(X_0, X_1) = (1, -\overline{p-1})$ we have a congruence identical in y, and the term y^0 gives the congruence

$$(-\overline{p-1})^{p^n} + \overline{p-1} \equiv 0 \quad [\mathfrak{M}],$$

whence follows the statement (2) in case $p = 2$ for $g = 1$ at once, and in case $p > 2$ for $g = (\overline{p-1}^{p^n} - \overline{p-1})/p$ by the remark that $pg \equiv -p \; [p^2]$ and so $g \equiv -1 \; [p]$.

(3) $$p^{p^{n-1}} \equiv 0 \quad [\mathfrak{M}].$$

Proof. In case $p = 2$ (3) follows at once from (2), $2 \equiv 0 [\mathfrak{M}]$, as proved for $p = 2$. In case $p > 2$ the coefficient of $X_0^2 X_1^{p^n - 1}$ gives

$$[\tfrac{1}{2} p(p-1)]^{p^{n-1}} \cdot (1 + y + \cdots + y^{p-1}) \equiv 0 \quad [\mathfrak{M}].$$

Setting $G_1 = \frac{1}{2}(p-1)$ and $G_2 = 1 + \cdots + y^{n-1}$ we have (1) quantities G_1', G_2' of $\mathfrak{R}_1$ for which $G_1 G_1' \equiv 1$, $G_2 G_2' \equiv 1$ $[\mathfrak{M}]$, and obtain (3) in multiplying the congruence just written by $G_1'^{p^{n-1}} G_2'$.

(4) $$p \equiv 0 \quad [\mathfrak{M}].$$

Proof. From (2), $[p, g] \sim [1]$, it follows by § 3 (3) that $[p^{p^{n-1}}, g] \sim [1]$, that is, for properly determined g', $gg' \equiv 1$ $[p^{p^{n-1}}]$, so that by (3) $gg' \equiv 1$ $[\mathfrak{M}]$, whence from (2), $pg \equiv 0$ $[\mathfrak{M}]$ we have the desired congruence (4).

(5) $$y^{p^n} - y \equiv 0 \quad [\mathfrak{M}].$$

Proof. (5) follows from the identical congruence by the substitution $(X_0, X_1) = (1, y)$ with the remark that, since $P[X_0, X_1]$ has the factor $X_1 + \overline{p-1} y X_0$, $P[1, y] \equiv 0$ $[p]$ and so by (4) $P[1, y] \equiv 0$ $[\mathfrak{M}]$.

10. We need further the known decomposition

$$[p, y^{p^n} - y] \sim \prod_d \prod_{j=1, m_d} [p, F_{dj}[y]],$$

where for every divisor d of n the $F_{dj}[y] (j = 1, \cdots, m_d)$ are the say m_d quantities of $\mathfrak{R}_1$ congruentially distinct and each of degree d and irreducible modulo p. We may and do suppose that the coefficients of the $F_{dj}[y]$ are taken from the integers $0, 1, \cdots, p-1$. Here the $F_{nj}[y]$ are the $F_{nj}[y]$ of § 4; $m_n = m$. The systems $[p, F_{dj}[y]]$ are by pairs relatively prime.

11. Proof of (B, C): $\mathfrak{M} \sim \mathfrak{F}_n \sim \prod_{j=1, m} \mathfrak{F}_{nj}$. We have from § 9 (4, 5)

$$\mathfrak{M} \sim [\mathfrak{M}, p, y^{p^n} - y],$$

whence by § 10 and § 3 (1) we have the decomposition

$$\mathfrak{M} \sim \prod_d \prod_{j=1, m_d} [\mathfrak{M}, p, F_{dj}[y]],$$

and so, since $[\mathfrak{M}, F_{dj}[y]] \sim [1]$ for $d < n$ (§ 9, 1),

$$\mathfrak{M} \sim \prod_{j=1, m} \{\mathfrak{M}, p, F_{nj}[y]\},$$

that is, in the notation of § 4,

$$\mathfrak{M} \sim \prod_{j=1,\, m} \cdot [\mathfrak{M}, \mathfrak{F}_{nj}].$$

Now by (A'') $\mathfrak{M} \equiv 0$ $[\mathfrak{F}_{nj}]$ and so $[\mathfrak{M}, \mathfrak{F}_{nj}] \sim \mathfrak{F}_{nj}$, and thus we have finally

$$(B,C) \quad \mathfrak{F} \sim \prod_{j=1,\, m} \mathfrak{F}_{nj} \sim \mathfrak{F}_n.$$

12. It is noteworthy that the modular system $\mathfrak{M} = \mathfrak{M}_{k+1,\, n,\, p}$ is in the sense of equivalence in fact independent of k and dependent only on n and p,

$$\mathfrak{M}_{k+1,\, n,\, p} \sim \mathfrak{F}_n \sim \mathfrak{M}_{2,\, n,\, p}.$$

13. My exhibition of the system $\mathfrak{F}_n$ by its equivalent $\mathfrak{M}_{k+1,\, n,\, p}$ for k any positive integer, the elements of $\mathfrak{M}_{k+1,\, n,\, p}$ being the coefficients of the integral function $M_{k+1,\, n,\, p}$, is to be compared with that of Serret (Algèbre supérieure, fifth edition, volume 2, §349). In our notation we have (using §3, 1°)

$$\mathfrak{F}_n \sim \prod_{j=1,\, m} \mathfrak{F}_{nj} \sim \prod_{j=1,\, m} [p, F_{nj}[y]] \sim [p, F_n[y]],$$

where

$$F_n[y] = \prod_{j=1,\, m} F_{nj}[y].$$

Serret gives a fraction $F[y]$

$$\prod_{n_e} (y^{p^{n_e}} - y) / \prod_{n_o} (y^{p^{n_o}} - y),$$

where n_e, n_o run through those divisors of n whose complementary divisors n/n_e, n/n_o have respectively an even (or 0), an odd number of unrepeated prime factors, and shows that modulo p the division indicated by the notation of $F(y)$ can be performed and that for the resulting integral function $F[y]$ we have $F[y] \equiv F_n[y]$ $[p]$ and so $\mathfrak{F}_n \sim [p, F[y]]$. — Apparently my exhibition lends itself more easily to investigations in the domain of pure arithmetic.

THE UNIVERSITY OF CHICAGO.

SYSTEMS OF EXTREMALS IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, December 29, 1906.)

THE extremals connected with any problem in the calculus of variations are the curves or surfaces or higher manifolds which satisfy the first of the conditions necessary for a maximum or minimum value. In the case of the integral of simplest type

$$\int F(x, y, y')dx \tag{1}$$

this condition is given by the Euler-Lagrange equation

$$\frac{d}{dx}F_{y'} - F_y = 0, \tag{2}$$

where the subscripts indicate partial differentiation. The expanded equation is

$$F_{y'y'}y'' + F_{y'y}y' + F_{y'x} - F_y = 0. \tag{3}$$

The extremals here form a doubly infinite system of plane curves. Darboux's well-known investigation of the inverse problem shows that the system is entirely general*: Any doubly infinite system of curves may be regarded as the system of extremals belonging to some integral of type (1).

In this respect the simplest type of problem is unique. In all the more complicated integrals, involving either derivatives of higher order, or more than one dependent or independent variable, the Euler-Lagrange equations are of special form, and hence the system of extremals (curves or other manifolds) must possess peculiar geometric properties. Our object is to give an example, apparently the simplest, of such a property.

Consider the plane problem of second order connected with the integral

$$\int F(x, y, y', y'')dx. \tag{4}$$

* Darboux, Théorie des surfaces, vol. 3, no. 604.

The extremals form a quadruply infinite system of curves with the equation

$$\frac{d^2}{dx^2} F_{y''} - \frac{d}{dx} F_{y'} + F_y = 0. \tag{5}$$

This is found to be of the form

$$y^{IV} = Ay'''^2 + By''' + C, \tag{6}$$

where A, B, C involve only x, y, y', y''. It will not be necessary, for our purpose, to have the values of these coefficients in terms of the partial derivatives (of first, second, and third order) of F.

An extremal is determined when the initial values of x, y, y', y'', y''' are assigned. If only x, y, y', y'' are given, that is, a curvature element, there will be ∞^1 corresponding curves. For each of these at the common point construct the osculating conic. The locus of the centers of the conics so obtained may be termed the *central locus* corresponding to the given curvature element. What is the character of the central locus?

The general formulas for the center of the conic osculating a given curve at a given point may be obtained most readily by means of a theorem of Cayley: the center is determined by the intersection of consecutive axes of deviation.* Using coordinates X, Y referring to the given point as origin, we find the equation of the axis of deviation at the given point to be

$$Y/X = m, \tag{7}$$

where

$$m = \frac{y'y''' - 3y''^2}{y'''}. \tag{7'}$$

At the consecutive point the equation becomes

$$\frac{Y - dy}{X - dx} = m + m'dx, \tag{8}$$

where

$$m' = \frac{y''(3y''y^{IV} - 5y'''^2)}{y'''^2}. \tag{8'}$$

*Salmon-Fiedler, Höhere Kurven, p. 468. The point in question is sometimes termed the center of deviation.

The intersection of (7) and (8) is the required center

$$(9) \qquad X=\frac{3y''y'''}{5y'''^2-3y''y^{IV}}, \qquad Y=\frac{3y''(y'y'''-3y''^2)}{5y'''^2-3y''y^{IV}}.$$

To apply these general formulas to the question proposed, we first solve them with respect to y''', y^{IV}, obtaining

$$(10) \qquad y'''=\frac{-3y''^2X}{Y-y'X}, \quad y^{IV}=\frac{3y''^2(5y''^2X+Y-y'X)}{(Y-y'X)^2}.$$

Substituting these values in (6), we find the relation

$$(11) \qquad \begin{aligned} &3y''^2(Y-y'X) \\ &=3y''^3(3y''A-5)X^2-3y''^2BX(Y-y'X)+C(Y-y'X)^2. \end{aligned}$$

Here A, B, C have fixed values since the initial element x, y, y', y'' is supposed to be given. The central locus, defined by (11), is thus a conic. Furthermore it passes through the given point, the origin of the X, Y system; and its tangent at that point, namely,

$$(12) \qquad Y-y'X=0,$$

has the direction of the given element. We state the result as follows:

The extremals connected with any plane problem of the second order, that is, with any integral of type (4), *form a quadruply infinite system of curves such that* ∞^1 *are determined by a given element of curvature. The locus of the centers of the conics which osculate these* ∞^1 *curves at their common point is a conic tangent to the given element.*

From the discussion it is seen that the result is not completely characteristic since it holds for all systems whose differential equation is of form (6), and such a system need not be a system of extremals. In the latter case the functions A, B, C are connected by rather complicated relations derived by expressing the fact that certain partial differential equations are consistent.

That the property stated is, however, characteristic of the form (6) may be shown as follows. By assumption, the central locus for any element of curvature is to be a conic touching the

element. Its equation is then

$$Y - y'X = aX^2 + bXY + cY^2. \tag{13}$$

Substituting the general values given in (9), and reducing, we find that y^{IV} is quadratic in y'''. The equation is therefore of the form (6).

If it is required merely that the central locus shall be a conic, without any restriction as to position, then a more general class of differential equations of the fourth order is obtained, involving five instead of three arbitrary functions of x, y, y', y''. In particular, if the conic is to pass through the given point, the resulting equation is of the form

$$y^{IV} = ay'''^3 + \beta y'''^2 + \gamma y''' + \delta. \tag{14}$$

An interesting special case of systems of type (6) is obtained by imposing the condition that for every curvature element the conic (11) shall degenerate. This is so only when A has the special form

$$A = \frac{5}{3y''}. \tag{15}$$

The conic then breaks up into the trivial part (12) and the proper part

$$3y''^2(BX + 1) - C(Y - y'X) = 0. \tag{16}$$

In the case of a system of extremals the value of A in terms of the integrand function F is

$$A = -\frac{F_{y''\,y''\,y''}}{F_{y''\,y''}}. \tag{17}$$

By comparing (15) and (17) and integrating, we find

$$F = f_1(x, y, y')y''^{\frac{1}{3}} + f_2(x, y, y')y'' + f_3(x, y, y'), \tag{18}$$

where the f's are arbitrary functions.

Hence the only case in which the system of extremals connected with an integral (4) *is such that the central loci are straight lines is that in which the integrand function is of the form* (18).

COLUMBIA UNIVERSITY.

A NECESSARY CONDITION FOR AN EXTREMUM OF A DOUBLE INTEGRAL.

BY PROFESSOR MAX MASON.

(Read before the American Mathematical Society, April 28, 1906.)

LEGENDRE'S necessary condition on a function $y = \phi(x)$ which minimizes or maximizes the integral

$$\int_{x_0}^{x_1} f(x, y, y')dx$$

is that the function $f_{y'y'}(x, \phi, \phi')$ must be positive or zero for a minimum, negative or zero for a maximum. There seems to have been no adequate treatment of double integrals in regard to the analogous condition.

Suppose the function $z = \phi(x, y)$ is of class * D' in a region Ω, assumes the values g on the boundary of Ω, and renders the integral

$$J = \int\int_{\Omega} f(x, y, z, p, q)dxdy \quad (p = z_x, q = z_y)$$

an extremum with respect to all functions of class D' which assume the values g on the boundary. Let $\zeta(x, y)$ be an allowed variation, *i. e.*, a function of class D' which vanishes on the boundary. Then the value of the integral

$$J(\epsilon) = \int\int_{\Omega} f(x, y, \phi + \epsilon\zeta, \phi_x + \epsilon\zeta_x, \phi_y + \epsilon\zeta_y)dxdy,$$

where ϵ is a parameter, is a function of ϵ which has an extreme value for $\epsilon = 0$. It follows † that $J'(0)$ must vanish, and that $J''(0)$ must have the same sign or vanish, for all allowed variations ζ. From the condition $J'(0) = 0$ is derived the dif-

* A function u is of class $C^{(n)}$ in Ω when it is continuous within Ω and on the boundary, together with its derivatives of all orders less than or equal to n. If u is continuous throughout Ω, and if Ω may be divided into a finite number of sub-regions such that u is of class $C^{(n)}$ in each closed sub-region, then u is of class $D^{(n)}$ in Ω. This nótation is due to Bolza.

† The derivatives $J'(\varepsilon)$ and $J''(\varepsilon)$ are continuous, under the assumption that f is of class C''.

ferential equation which ϕ must satisfy. The second condition states that the integral

$$\delta^2 J = \int\int_\Omega \{\zeta_x^2 f_{pp} + 2\zeta_x \zeta_y f_{pq} + \zeta_y^2 f_{qq} + \zeta^2 f_{zz} + 2\zeta\zeta_x f_{zp} + 2\zeta\zeta_y f_{zq}\}\, dxdy$$

must have the same sign, or vanish, for all allowed variations; the arguments of the derivatives of f are $x, y, \phi, \phi_x, \phi_y$. It has been shown * that if

$$\alpha^2 f_{pp} + 2\alpha\beta f_{pq} + \beta^2 f_{qq}$$

is a positive definite form $\delta^2 J$ is positive, if a negative definite form $\delta^2 J$ is negative. In point of fact it is a necessary condition for either a minimum or a maximum that this form be at least semi-definite. This will be proved without transforming the second variation, $\delta^2 J$.†

The theorem to be proved is then the following:

A necessary condition for an extremum is that

$$f_{pp} f_{qq} - f_{pq}^2 \geqq 0 \tag{1}$$

at all points of Ω.

Suppose, to the contrary, that the first member of (1) is negative at a point (x_0, y_0) of Ω. Then a region R containing the point (x_0, y_0) may be chosen, so small that throughout R ϕ_x and ϕ_y are continuous and

$$f_{pp} f_{qq} - f_{pq}^2 < -m^2, \tag{2}$$

where m is a fixed number. There exist, therefore, two real and distinct roots of the equation

$$Q(h) \equiv f_{pp} h^2 + 2 f_{pq} h + f_{qq} = 0$$

for each point of R. $Q(h)$ will have different signs according as h lies between or outside these roots. Since the coefficients of $Q(h)$ are continuous within R, it follows that if R be chosen

*See Kneser, Encyklopädie der Mathematischen Wissenschaften, II A8, § 25.

† The necessity of Legendre's condition in the case of simple integrals could be shown in a similar manner, without transforming the second variation.

small enough, two fixed intervals P and N and a fixed number δ may be found, such that at all points of R

$$(3)\qquad \begin{aligned} f_{pp}h^2 + 2f_{pq}h + f_{qq} &> \quad \delta^2 \text{ for all } h \text{ in } P, \\ f_{pp}h^2 + 2f_{pq}h + f_{qq} &< -\delta^2 \text{ for all } h \text{ in } N. \end{aligned}$$

Choose for the region R a rhombus, whose center is at (x_0, y_0) and whose sides are formed by the lines

$$(a)\quad u_1 - \epsilon = 0, \qquad (c)\quad u_1 + \epsilon = 0,$$

$$(b)\quad u_2 - \epsilon = 0, \qquad (d)\quad u_2 + \epsilon = 0,$$

where

$$u_1 = (x - x_0)\cos\alpha_1 + (y - y_0)\sin\alpha_1,$$

$$u_2 = (x - x_0)\cos\alpha_2 + (y - y_0)\sin\alpha_2,$$

the positive constant ϵ being chosen so small that (3) holds in R. The angles α_1, α_2 will be determined later. The diagonals of the rhombus divide it into four triangles α, β, γ, δ, containing the sides a, b, c, d, respectively. An allowed variation ζ is defined by the following equations, ζ being zero outside of R:

$$\zeta = -u_1 + \epsilon \text{ in } \alpha, \qquad \zeta = u_1 + \epsilon \text{ in } \gamma,$$

$$\zeta = -u_2 + \epsilon \text{ in } \beta, \qquad \zeta = u_2 + \epsilon \text{ in } \delta.$$

Now

$$\delta^2 J = \delta_1^2 J + \delta_2^2 J,$$

where

$$\delta_1^2 J = \int\int_R \zeta_y^2 \left\{ \left(\frac{\zeta_x}{\zeta_y} \right)^2 f_{pp} + 2\frac{\zeta_x}{\zeta_y} f_{pq} + f_{qq} \right\} dxdy,$$

$$\delta_2^2 J = \int\int_R \{\zeta^2 f_{zz} + 2\zeta\zeta_x f_{zp} + 2\zeta\zeta_y f_{zq}\} dxdy.$$

Since $|\zeta_x| \leqq |1|$, $|\zeta_y| \leqq 1$, $|\zeta| \leqq \epsilon$, it follows that

$$(4)\qquad |\delta_2^2 J| \leqq \epsilon MA(\epsilon + 4),$$

where M is the greatest of the maxima of $|f_{zz}|$, $|f_{zp}|$, $|f_{zq}|$, in R, and A is the area of R.

Now choose the angles α_1 and α_2 so that both of the quantities cot α_1, cot α_2 lie either in the interval P or in the interval N. Then the value of ζ_x/ζ_y will lie either in P or in N; if in P $\delta_1^2 J$ will be positive, if in N $\delta_1^2 J$ will be negative, on account of (3). In either case

$$|\delta_1^2 J| > \delta^2 \int\int_R \zeta_y^2 dx dy = \delta^2 \frac{A}{2}(\sin^2 \alpha_1 + \sin^2 \alpha_2),$$

where A is the area of R. Therefore, from (4),

$$\delta^2 J > A\left[\frac{\delta^2}{2}(\sin^2 \alpha_1 + \sin^2 \alpha_2) - \epsilon M(\epsilon + 4)\right]$$

if cot α_1 and cot α_2 lie in P, and

$$\delta^2 J < A\left[-\frac{\delta^2}{2}(\sin^2 \alpha, + \sin^2 \alpha_2) + \epsilon M(\epsilon + 4)\right]$$

if cot α_1 and cot α_2 lie in N. If ϵ be taken sufficiently small, $\delta^2 J$ may therefore be made positive or negative at pleasure. This result, obtained under the supposition that the inequality (1) is not satisfied at some point in Ω, contradicts the assumption that the function ϕ makes the integral J an extremum. *The condition* (1) *is therefore necessary for an extremum.*

The above proof of the necessity of condition (1) depends upon a variation whose first derivatives have finite discontinuities. If the extremum problem be so stated that the functions z are required to satisfy more restrictive conditions of continuity the proof must be altered, since the variations must satisfy these conditions also.

Suppose * the functions z are required to be of class C''. For the region R, chosen in the previous case as a rhombus, take the region formed by the sub-regions R_1, R_2, R_3, defined in the following manner: R_1 is a rectangle having the sides

$$u = \epsilon, \quad v = d, \quad u = -\epsilon, \quad v = -d,$$

where

$$u = (x - x_0)\cos\alpha + (y - y_0)\sin\alpha,$$

$$v = (x - x_0)\sin\alpha - (y - y_0)\cos\alpha.$$

* This is the assumption usually made in dealing with double integrals.

The regions R_2 and R_3 are the portions exterior to R_1 of the areas bounded by the closed curves

$$u^2 + (v-d)^4 - \epsilon^2 = 0, \qquad u^2 + (v+d)^4 - \epsilon^2 = 0,$$

respectively.

A variation ζ of class C'' is defined by the following equations, ζ being zero outside of R:

$$\zeta = (u^2 - \epsilon^2)^4 \qquad \text{in } R_1,$$

$$\zeta = (u^2 + (v-d)^4 - \epsilon^2)^4 \qquad \text{in } R_2,$$

$$\zeta = (u^2 + (v+d)^4 - \epsilon^2)^4 \qquad \text{in } R_3.$$

Now

$$\delta^2 J = \int\int_{R_1} \zeta_y^2 \left\{ \left(\frac{\zeta_x}{\zeta_y}\right) f_{pp} + 2\frac{\zeta_x}{\zeta_y} f_{pq} + f_{qq} \right\} dxdy$$
$$+ \int\int_{R_1+R_2} \{ \zeta_x^2 f_{pp} + 2\zeta_x\zeta_y f_{pq} + \zeta_y^2 f_{qq} \}\, dxdy$$
$$+ \int\int_R \{ \zeta^2 f_{zz} + 2\zeta\zeta_x f_{xp} + 2\zeta\zeta_y f_{zq} \}\, dxdy.$$

Since $|u| \leqq \epsilon$ in R_1, $|v-d| \leqq \sqrt{\epsilon}$ in R_2 and $|v+d| \leqq \sqrt{\epsilon}$ in R_3, it follows that at all points of R, $|\zeta_x| < const.\ \epsilon^7$, $|\zeta_y| < const.\ \epsilon^7$. Throughout R, $|\zeta| \leqq \epsilon^8$. The second and third integrands in the second member of the above equation are therefore numerically less than $const.\ \epsilon^{14}$ and $const.\ \epsilon^{15}$ respectively. The area of R is less than $4(d + \sqrt{\epsilon})\epsilon$, and the area of $R_2 + R_3$ is less than $4\epsilon^{\frac{3}{2}}$. Hence, for all values of ϵ sufficiently small,

$$\left| \delta^2 J - \int\int_{R_1} \zeta_y^2 \left\{ \left(\frac{\zeta_x}{\zeta_y}\right)^2 f_{pp} + 2\frac{\zeta_x}{\zeta_y} f_{pq} + f_{qq} \right\} dxdy \right| < K.\epsilon^{\frac{31}{2}},$$

where K is a constant. Now in R_1

$$\zeta_x/\zeta_y = \cot\alpha.$$

Therefore, if $\cot\alpha$ lies in the interval P, the integral in the first member of the above inequality is positive; if $\cot\alpha$ lies in N, it is negative. In either case, on account of (3), its

absolute value is greater than

$$\delta^2 \int\int_{R_1} \zeta_y^2 dx dy = 64\ \delta^2 \sin^2 a \int_{-d}^{d} \int_{-\epsilon}^{\epsilon} (u^2 - \epsilon^2)^6 u^2 du dv = c^2 \epsilon^{15},$$

where c is a constant independent of ϵ.

Therefore, if cot α lies in the interval P,

$$\delta^2 J > \epsilon^{15}(c^2 - K\sqrt{\epsilon}).$$

If cot α lies in the interval N,

$$\delta^2 J < \epsilon^{15}(-c^2 + K\sqrt{\epsilon}).$$

If ϵ be taken sufficiently small $\delta^2 J$ may therefore be made positive or negative at pleasure, and the necessity of condition (1) follows at once.

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SHORTER NOTICES.

Handbuch der Theorie der Gammafunktion. By Dr. NIELS NIELSEN, Docent in Mathematics at the University of Copenhagen. Leipzig, Teubner, 1906. 326 pages.

As the name indicates, this book is a comprehensive handbook of the theory of the gamma function; it contains nearly all the known formulas, gives very exhaustive references, and arranges the entire material into a systematic and readable whole. Besides a full index, a bibliography of over five hundred titles is appended to the volume, with exact citation including date and number of pages of each memoir. That Dr. Nielsen is fitted for the task which he had set himself is attested by the long list of his own contributions, exceeded only by those of Euler, Lerch and Schlömilch.

The book is divided into three parts: the first treats of series and the analytic theory, the second of the theory of the definite integral, and the third of the inverse problem of expanding other series in terms of gamma functions.

The first part (112 pages, 581 numbered formulas) com-

mences with the difference equation having $\Gamma(x)$ for a particular solution; Euler's constant is derived, and $\Gamma(x)$, $B(x, \frac{1}{2})$ defined so as to apply to irrational and complex values of the arguments. The functions $P_a(x)$, $Q_a(x)$ are then introduced and similarly discussed. The various functions are all developed into series, a chapter on the evaluation of infinite series and infinite products being added. The Stirling numbers and polynomials receive a whole chapter for their treatment. The functions previously defined are then expressed in terms of them.

A short chapter treats of zero points and of limiting values for infinite values of the argument. This part closes with Hölder's theorem that no algebraic differential equation can have $\Gamma(x)$ for a solution, but the proof is so given that the theorem applies at once to a number of related functions. Mention is occasionally made of gaps in the theory that have not been bridged, as, *e. g.*, the nature of Euler's constant, the roots of $\beta(x)$, the convergence of certain series, etc.

The second part (122 pages, 563 numbered formulas) begins with a very general consideration of functional equations which include the gamma function and the related functions as particular cases. The work is so general and so complete as to make it unsuitable for a textbook, though all the more valuable as a handbook. In the discussion of integration around a contour in the complex plane, the value of the book would have been enhanced and the volume made more easy of ready reference, had the path of integration been indicated by a figure. While the exact form is given in the text in each case, it is frequently necessary to hunt carefully for it; a reader using a formula for reference might easily mistake the path, understanding an ordinary rectilinear integration. The treatment of the asymptotic series, known as Stirling's series, is far too meagre, though no attempt is made at numerical evaluation anywhere in the book. The converging series involving Euler's constant is of little use for numerical purposes, while the asymptotic series can, by use of the functional equation, be used to determine the value of $\Gamma(x)$ to any number of places.

The third part (64 pages, 271 numbered formulas) is concerned with the inverse problem of expressing certain other functions in terms of $\Gamma(x)$ as function of development. This part is essentially the work of the author; its main accomplishment is the rigorous establishment of various results given by Stirling, and the pointing out that certain series derived formally

by Gauss and others are not convergent within the region considered. A number of functions not expressible in convergent series have been defined asymptotically. Skillful manipulations of sums and products of series are given, and the possible region of continuation defined. Finally, these various facultative series are applied to the gamma function itself, and the various related functions.

The typographical work is excellent. A few misprints have crept in, some of which might lead to confusion when the book is used for reference. I mention the following:

Page 31, line 8 (from bottom), for 3 read 2, 3.
" 41, " 12 " " " Jefferey " Jeffery.
" 52, " 2 " " " $\psi\left(\frac{x+1}{2}+C\right)$ read $\psi\left(\frac{x+1}{2}\right)+C.$
" 58, equation (4), for q read s.
" 61, " (10), " $a_0 x^{n'}$ " $a_0 x^n$.
" 62, " (12), " (cotg " cotg (.
" 65, " (11), " $e^{+\frac{y}{x+s}}$ " $e^{-\frac{y}{x+s}}$.
" 103, line 3, insert Differential before Gleichung.
" 106, equation (7), for ϕ_n^{an} read $\phi_n^{a_n}$.
" 120, " (3), " $\psi(t)$ " $\phi(t)$.
" 129, " (5), " cs θ " cos θ.
" 131, " (2), " s " s.
" 136, " (17), insert dt.
" 144, line 1, for e^- read e^{-t}.
" 230, equation (3), for t read z.
" 231, " (8), " t " z.
" 233, " (13), insert dt and lower limit 0.
" 234, " (19), for $s=0$ read 0.
" 257, " (13), " $x=1$ " $x-1$.
" 265, " (3), " $\infty \cdot \frac{1}{\alpha}$ " ∞.
" 268, line 5, for füt read für.
" 279, equation (5), for ω read n.
" 282, " (7), insert f after first integral sign.
" 285, " (1), " -1 in brackets, $x-1$ is exponent of t.
" 288, in the formula, insert dt after sign of integration.

VIRGIL SNYDER.

Mélanges de Géométrie à Quatre Dimensions. Par E. JOUFFRET. Paris, Gauthier-Villars, 1906. 227 pp.

IN the preface to this work the author expresses its purpose as follows: "Dans l'avant-propos de notre traité élémentaire de géométrie à quatre dimensions (page xviii), nous avons avancé que les théories classiques peuvent trouver d'utiles inspirations dans ce quatrième champ que nous avons appellé l'étendue, monde géométrique plus vaste que le leur, où des droites, des plans et des espaces sont déterminés respectivement par deux, trois, et quatre points, et sont coupés par un espace suivant des points, des droites, et des plans. Nous nous proposons de donner maintenant quelques applications de nature à justifier cette proposition."

The book contains eight chapters bearing the following titles: Coup d'œil sur les principes de la géometrie à quatre dimensions; Le système de coordonnées et les trois premiers polyèdres réguliers; L'hexagramme de Pascal; La surface du troisième degré; L'hexagramme et l'hexastigme; Les hypersurfaces du second degré; Les quartiques; La question de l'existence réelle de l'hyperespace. Few errata have been noted by the reviewer, and generally they are of such a nature as not to occasion any difficulty. A curious oversight occurs on page 8, line 4, where each of the equations $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$ is spoken of as defining a plane. On the preceding page it is expressly stated that, in the geometry of four dimensions, a plane is defined by two linear relations among the variables. Each chapter closes with a brief summary giving in a concise form the results that have been obtained. In chapters involving many details these summaries are very useful. In Chapter VIII arguments are drawn from various sources in support of the existence of hyperspace. The author states that these are given because they are of interest, although the question does not enter in the field of mathematics. All that the author desires is simply to establish the utility of the idea.

One might be tempted to criticize the book on the ground of lack of unity. The author first devotes his attention to a discussion of the system of coordinates in the geometry of four dimensions; he then inserts two chapters, one dealing with a problem in the plane and the other dealing with a problem in space, before he comes back to the four dimensional geometry.

However, he has chosen a title for his book which permits considerable freedom both as to the matter presented and as to the order of presentation.

PETER FIELD.

Neuere Darstellungen der Grundprobleme der reinen Mathematik im Bereiche der Mittelschule. By ALOIS LANNER. Berlin, Otto Salle, 1907. viii + 192 pp. Price, 3 Marks.

ONE of the distinctive features of the teaching of elementary mathematics in the latter part of the nineteenth century was the influx of a large body of new theory relating to the fundamental laws underlying the common processes. Perhaps the best evidence of this is found in the lectures of Weierstrass, and among the best exponents of the movement is Stolz's "Allgemeine Arithmetik." It was natural to expect, however, that Weierstrass would never directly reach the teaching body of Germany, and that works as elaborate as those of Stolz would have but little practical influence in the schools.

The opening of the twentieth century is seeing an effort to bring the results of such labors as these to the attention of those who teach the elements. Naturally this involves a great deal of experiment. The college professor, with little knowledge of the powers and interests and immediate needs of preparatory students, is liable to insist upon secure foundations for every process, while the teacher in the classroom is equally likely to err the other way. For those who try to see the argument of each of these types, and to weigh them judicially, any effort to simplify the labors of the theorists and to present them in concise form, is very welcome.

Dr. Lanner has attempted exactly the work. He has not written a textbook, nor a work on the theory of teaching, but he has prepared a simple treatise that seeks to supplement each. In brief, it may be described as a handbook, giving in simple form the principles underlying each of the chapters of arithmetic and algebra as taken up in the elementary courses, and offering material for supplementing the theory of the textbook.

The general nature of the work can best be judged by a few of the chapter heads: Gleicheit und Grösse; Die natürlichen Zahlen, followed by the various operations; Erweiterung des Zahlengebietes durch die Subtraktion, introducing negative numbers; Die Teilbarkeit der ganzen Zahlen, both absolute

and relative; Die systematische Darstellung der ganzen Zahlen, with the operations; Erweiterung des Zahlengebietes durch die Division, followed by the various forms of fractions; Das Rechnen mit endlichen Grenzwerten, with the relation between the decimal and common fraction; Irrationale Zahlen und Grenzwerte, unfortunately little more than a title; Das Rechnen mit algebraischen Funktionen; Die Veränderlichkeit der Funktionen, with a brief treatment of the differential coefficient and with a hint at maxima and minima; Die Integration, a mere introduction of four pages; Die Verhältnisse, followed by proportion; Die Teilregel, Mischungs- und Gesellschaftsrechnung, a subject that seems never to lose its fascination with the German arithmetician; Prozentrechnung, Rabatt, Diskonto, und Terminrechnung; Regeldetri und Kettensatz; Die Gleichung ersten Grades mit einer Unbekannten, followed by two unknowns; Rechnungsoperationen dritter Stufe, *i. e.*, powers, roots, and irrational equations; Die Logarithmen, with exponential equations; Die Ableitung von Funktionen aus ihren Eigenschaften; Imaginäre und komplexe Zahlen; Quadratische Gleichungen, followed by indeterminate equations, series, and interest computation; Kombinatorik, followed by the binomial theorem; Politische Arithmetik, including probabilities and insurance. Doubtless under the influence of that annoying parsimony which frequently characterizes the German publisher, the book has neither table of contents nor index, and hence it loses much of the value that it might possess as a handy tool.

From the list given above it will be seen that, while the theory of arithmetic is fairly covered, with the exception of the irrational number, the supplementary material in algebra is less satisfactory. Indeed, in spite of the fact that the author seeks to establish solid foundations for algebra, the result at once strikes the reader as incomplete. Possibly this is necessary, considering the purpose he had in mind, but it raises the question as to whether the work is worth the doing in this manner.

In his numerous historical notes Dr. Lanner is not altogether fortunate. Harriot, for instance, was not a professor at Oxford, nor did he die in 1622; it is not known that Recorde was born in 1510, nor that Paciuolo died in 1514; Newton was not born in 1643; Lagrange was not working in Turin in any of the years when he is said to have been there, and Jordanus Nemorarius probably did not die in Rome. The prac-

tice of placing a date after a man's name is helpful when it has any definite significance, but with Dr. Lanner it has none. Sometimes he gives the date of death, as of Stevin (1620); sometimes the date of a book, as in the case of Vieta (1591); but quite as often the date has no particular significance, as in the cases of Tartaglia (1557) and Riese (1550). Usually, however, the important date is omitted entirely, as with Stevin's work on decimals (1585). There are also several unfortunate errors in technical and proper names, as in the cases of Ouchtred, Boëthius and Boethius, Goss (for Coss), and Wimburgh (for Edinburgh).

Altogether, therefore, the work may be said to have been written with a laudable purpose, but to leave the field open for a carefully prepared treatise on the same subject.

DAVID EUGENE SMITH.

Reformvorschläge für den mathematischen und naturwissenschaftlichen Unterricht; entworfen von der Unterrichtskommission der Gesellschaft deutscher Naturforscher und Aerzte. Zweiter Teil: *Vorschläge überreicht der 78 Naturforscherversammlung in Stuttgart, 1906.* Herausgegeben von A. GUTZMER, Leipzig, Teubner, 1906. Pp. 73.

THE work of the commission appointed by the German society of natural scientists and physicians, as summarized in this second annual report, has been carried on during the past year along the lines laid down in its first report which has already been reviewed in the BULLETIN * So far as mathematics is concerned, the present report consists of a discussion of the way in which the principles laid down in the first report would work out in practice in certain special types of German schools not there considered, and brings little of general significance beyond the boundaries of Germany.

The section on instruction in mathematics in the higher schools for girls is of some interest to us in America, where thousands of girls and young women annually carry through successfully work of a grade as advanced and as difficult as any that is given to the young men in the German gymnasia. The commission urges that more work in mathematics be given in the schools for girls, and, indeed, that those who complete the full course should end with substantially the same attainments

* Vol. 12, (1906), pp. 347-351.

as the young men who have passed through a gymnasium. In the earlier years (to age fifteen or sixteen) the commission recommends somewhat more emphasis upon intuition and practical exercises, especially the making of models, than in corresponding work in the boys' schools, where the logical element must be brought more freely into play. The avoidance of pedantic proofs of statements that are intuitionally evident is particularly important in teaching girls, special emphasis should be laid on the esthetic side of mathematics, while the logical side should not be neglected; the instruction as a whole must have its own peculiar character, and be based upon textbooks specially prepared therefor.

In the later years of the course no special differences in the instruction of young men and women are recommended, since the very fact that the women go on into the later years is of itself sufficient evidence to the commission of a somewhat mannish type of mind that may reasonably be expected to prove equal to the logical demands made on it by the mathematics to be given, extending perhaps as far as the elements of analytic geometry.

J. W. A. YOUNG.

N.-H. Abel, sa Vie et son Oeuvre. Par CH. LUCAS DE PESLOÜAN. Paris, Gauthier-Villars, 1906. Pp. xiii + 168.

No pretension is made in this biography to add to the facts previously published relating to Abel. It will not supersede the biography written by C.-A. Bjerknes and brought out in 1885 by the same publishers. The aim of De Pesloüan is to put the story of the short life of the gifted Norwegian in attractive form for scientific readers who are not specialists along the lines of Abel's researches. The idea is a commendable one and should be carried out more frequently in biographies of scientists. The author exhibits great sympathy and admiration for Abel. In the preface he goes so far as to call Abel's researches "l' oeuvre du plus grand mathématicien du XIX^{e} siècle." More judicious and suggestive is a remark which is attributed by the author to Hermite (page 136), by some others to Sylvester: "Abel a laissé aux mathématiciens de quoi travailler pendant cent cinquante ans."

It is well known that Abel, in his twenty-third year, started

upon a journey to continue his studies and meet the mathematicians of his day, that this diffident and sensitive youth was cordially received by Crelle in Berlin, that he went to Paris without stopping on the way to see Gauss at Göttingen, who he imagined had slighted him by paying no attention to his first draft of the proof of the impossibility of an algebraic solution of the quintic, and that in Paris he failed to receive appreciation and stimulus from Cauchy, Dirichlet, Poisson, Legendre, Fourier, Arago, Ampère and others. An article which Abel submitted to the French academy, containing what is known now as "Abel's theorem" on "abelian functions," was lost sight of, and not published until twelve years after his death. De Pesloüan discusses at length the causes of this lack of recognition. He holds (page 130) that at this period French mathematicians had quite generally abandoned the cultivation of pure mathematics, to take up applications to heat, elasticity and electricity, that the failure to appreciate the importance of the paper was not, as charged in a sketch of Abel in Michaud's Biographical Dictionary, due to indifference, nor to egoism. "Si les savants furent coupables envers lui, ce fut par suite de leur ignorance ou plutôt de leur incompréhension."

When Abel was in Paris, Galois was sixteen years old and had not yet made his début in mathematics. Had the lives of these two men of genius been spared longer to science, what further revelations might they not have made!

De Pesloüan says, page 109, that Abel's second paper on "Recherches sur les fonctions elliptiques" is lost. He does not seem to be aware that it was found by G. Mittag-Leffler in 1894 and published in the *Acta Mathematica* (volume 26, pages 1–42).*

Some errors in the spelling of proper names and in the statement of theorems might have been avoided by greater care in the reading of the proofs.

FLORIAN CAJORI.

* G. Eneström in *Bibliotheca Mathematica*, ser. 3, vol. 7 (1906), page 217.

NOTES.

AT the meeting of the AMERICAN MATHEMATICAL SOCIETY on April 27, Professor W. F. OSGOOD will deliver his presidential address, postponed from the recent annual meeting. The subject of the address will be "The calculus in colleges and technical schools."

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY for 1907, containing the list of officers and members of the Society, constitution and by-laws, annual reports, and catalogue of journals in the Society's library together with other accessions for 1905–1906, has recently been issued. A complete catalogue of the library up to 1905 is contained in the Register for that year, copies of which may still be obtained from the Secretary.

THE opening (January) number of volume 8 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Generalization of the groups of genus zero," by G. A. MILLER; "On reflexive geometry," by F. MORLEY; "The groups in which every subgroup is either abelian or hamiltonian," by G. A. MILLER; "On modular groups isomorphic with a given linear group," by H. F. BLICHFELDT; "Denumerants of double differentiants," by W. E. STORY; "The groups of classes of congruent matrices, with application to the group of isomorphisms of any abelian group," by A. RANUM; "A theorem of Abel and its application to the development of a function in terms of Bessel's functions," by C. E. SMITH; "Irreducible linear homogeneous groups whose orders are powers of a prime," by W. B. FITE; "Applicable surfaces with asymptotic lines of one surface corresponding to a conjugate system of the other," by L. P. EISENHART.

AT the meeting of the London mathematical society held on January 10 the following papers were read: By A. STOTT, "Exhibition of four-dimensional models"; by E. W. HOBSON, "On the uniform convergence of Fourier series"; by J. E. LITTLEWOOD, "Asymptotic approximation of integral functions of zero order"; by A. R. FORSYTH, "Partial differential equations of the second order having integral systems free from

partial quadratures"; by G. H. HARDY, "On the singular points of some classes of power series in several variables"; by W. BURNSIDE, "The construction of the line drawn through a given point to meet two given lines"; by P. W. WOOD, "On the reduction of covariants of binary quantics of infinite order, part III"; by A. CUNNINGHAM, "On hyper-even numbers, and on Fermat's numbers."

THE annual meeting of the British mathematical association was held at King's College, London, January 26. Professor G. H. BRYAN was elected president for the ensuing year. The following papers were read: by H. H. HUDSON, "Diagrams of anemoids"; by A. LODGE, "On contracted methods in arithmetic"; by C. S. JACKSON, "On the elementary arithmetic of the theory of numbers." The association now includes 419 members.

AT the annual public meeting of the Paris academy of sciences, held on December 17, 1906, the following prizes were awarded for mathematical memoirs: Grand prize (fr. 3000) divided among H. PADÉ (fr. 1500), R. DE MONTESSUS (fr. 1000) and M. AURIC (fr. 500); Francoeur prize (fr. 1000) to E. LEMOINE; Poncelet prize (fr. 2000) to L. GUICHARD; Montyon prize (fr. 700) to G. MARIÉ; Lalande prize (fr. 540) divided between R. G. AITKEN and W. J. HUSSEY; Valz prize (fr. 460) to J. PALISA; Janssen medal to A. RICCÒ. The Wilde prize (fr. 4000) for general progress in science was awarded in part to M. MASSAU for his work on graphic integration; the Laplace prize (complete works of Laplace) to M. LÉVY. The subjects for which the prizes were offered were announced in the BULLETIN, volume 11, page 327, and volume 12, page 319.

The Academy announces the following prizes in mathematical sciences for 1908 and 1909; competing memoirs should be in the hands of the secretary before December 31 of the year preceding that in which the respective prizes are awarded:—The Grand prize and Francoeur prize as announced before, the Poncelet prize in 1908 for general progress in pure mathematics, in 1909 for general progress in applied mathematics; the Bordin prize (fr. 3000) for the following problem:

"The absolute invariant which represents the number of double integrals of the second kind on an algebraic surface depends upon a relative invariant ρ, which plays an important

part in the theory of integrals of total differentials of the third kind and in the theory of algebraic curves traced upon the surface. It is proposed to make an exhaustive study of this invariant and to determine in particular how its exact value can be found, at least for an extended category of surfaces" (competing memoirs must be received before January 1, 1909).

The Montyon, Fourneyron, Vaillant and Borleau prizes are to be awarded under the conditions previously announced.

COLUMBIA UNIVERSITY. — The following special courses of public lectures in mathematical physics are announced: — By Professor OTTO LUMMER, of the University of Breslau: The laws of radiation of the black body; their theoretical establishment and practical importance. Ten lectures, Fridays at 4 P. M., Saturdays at 10 A. M., beginning Friday, February 15. — By Professor JOSEPH LARMOR, of Cambridge University: Physical dynamics. Six lectures, Wednesday–Thursday, March 27–28; Friday–Saturday, April 5–6, April 12–13. The Wednesday and Friday lectures are given at 4 P. M., the Thursday and Saturday lectures at 10 A. M.

Detailed information regarding these courses is contained in the circulars issued by the University.

DR. R. BAIRE, associate in mathematics at the University of Dijon, has been promoted to a full professorship of mathematics at the same institution.

THE British royal astronomical society has awarded its gold medal for 1907 to Professor E. W. BROWN for his work on the motion of the moon.

PROFESSOR R. MÜLLER, of the technical school at Brunswick, has been appointed professor of mathematics at the technical school of Darmstadt.

PROFESSOR C. K. RUSSJAN, of the technical school at Lemberg, has been appointed professor of pure mathematics at the University of Charkow.

PROFESSOR K. BOEHM, of the University of Heidelberg, has been promoted to a full professorship, with special regard to elementary mathematics.

DR. E. W. HOBSON has been appointed chairman of the mathematical tripos, part II, for 1907.

PROFESSOR C. A. NOBLE, of the University of California, has been granted leave of absence and will spend next year in study at European universities.

DR. C. C. JONES has been appointed chancellor and professor of mathematics at the University of New Brunswick, to succeed the late Professor Harrison.

PROFESSOR G. W. JONES, of Cornell University, will retire from active service at the close of the present academic year with the title of professor emeritus.

AT the University of Texas, Professor H. Y. BENEDICT has been appointed professor of applied mathematics.

MR. G. A. ROSS, formerly of Hardin College, has been appointed instructor in mathematics at the George Washington University.

PROFESSOR E. JURGENS, of the technical school at Aachen, died January 5, 1907, at the age of 56 years. Professor A. MANNHEIM, professor emeritus at the Ecole Polytechnique of Paris, died December 11, 1906, at the age of 75 years. Professor J. A. OUDEMANS, of the University of Utrecht, died December 13, 1906, at the age of 78 years.

CATALOGUES of second-hand books: Martin Boas, Berlin, Karlstrasse 25, catalogue No. 60, 3082 titles in mathematics, physics and astronomy. — W. Heffer and Sons, Cambridge, Eng., catalogue No. 22, 1029 titles in mathematics and science, including the library of the late Professor Joly. — W. Junk, Berlin, Kurfürstendamm 201, 1629 titles in mathematics and science.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARNOUX (G.). Arithmétique graphique. Introduction à l'étude des fonctions arithmétiques. Paris, Gauthier-Villars, 1906. 8vo. 20 + 225 pp. Fr. 5.00

AUZEMBERGER. See VESSIOT (E.).

BERNSTEIN (R.). Les courbes du troisième degré. (Thèse.) Besançon, 1906.

BERTRAND (J.). Calcul des probabilités. 2e édition conforme à la 1re. Paris, Gauthier-Villars, 1907. 8vo. 57 + 322 pp. Fr. 12.00

CARVER (W. B.). On the Cayley-Veronese class of configurations. (Diss., Johns Hopkins.) Lancaster, New Era Printing Co., 1906. 8vo. 22 pp. and pp. 534–545.

FISKE (T. S.). Functions of a complex variable. New York, Wiley, 1906. 8vo. (Mathematical monograph series.) Cloth. $1.00

MALTBIE (W. H.). Analytic geometry: a first course. Baltimore, The Woman's College, 1906. 3 + 142 pp. $0.25

PAPELIER (G.). Précis de géométrie analytique, à l'usage des élèves de mathématiques spéciales, conforme au programme du 26 juillet 1904. Paris, Vuibert, 1907. 8vo. 304 pp.

RIOLLOT (J.). Les carrés magiques. Contribution à leur étude. Paris, Gauthier-Villars, 1907. 8vo. 124 pp. Fr. 5.00

SANTERRE (S.). Psychologie du nombre et des opérations élémentaires de l'arithmétique. La genèse des premières notions de l'arithmétique; notions de suite, de nombre, de somme, et de différence; produit et quotient. Avec une préface par P. Janet. Paris, Doin, 1907. 8vo. 14 + 182 pp.

STUYVAERT (M.). Les nombres positifs. Exposé des théories modernes de l'arithmétique élémentaire. Gand, van Goethem, 1906. 8vo. 133 pp. Fr. 3.00

TAFELMACHER (A.). Elementos de la geometría analítica del plano, para el uso del curso militar de la Escuela militar. Santiago de Chile, 1906. 8vo. 135 pp. P. 1.00

TANNENBERG (W. DE). Cours de géométrie analytique. Paris, Vuibert et Nony, 1906. 4to. 2 fascicules: pp. 1–88.

VESSIOT (E.). Leçons de géométrie supérieure professées à l'Université de Lyon. Rédigées par Auzemberger. Paris, Hermann, 1906. 4to. 326 pp. (Autogr.) Fr. 12.00

WHITEHEAD (A. N.). Axioms of projective geometry. Cambridge University Press, 1906. 8vo. 2s. 6d.

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BALLERINI (P.). La geometria pel ginnasio superiore e pel liceo. Monza, Artigianelli, 1906. 16mo. 246 pp. L. 2.00

BARBISCH (H.). See JAHNE (J.).

BAUDOIN (P.). See BOURLET (C.).

BERGMEISTER (H.). Geometrische Formenlehre für Mädchen-Lyzeen. Teil 2: Für die III. und IV. Klasse. 2te, umgearbeitete Auflage. Wien, Deuticke, 1906. 8vo. 4 + 69 pp. Boards. M. 1.40

BÖTTCHER (R.) und SENDLER (R.). Raumlehre für Lehrerseminare. Nach dem Lehrplan vom 1. VII. 1901 bearbeitet. Teil 1: Planimetrie. 2te Auflage. Breslau, Handel, 1907. 8vo. 4 + 106 pp. Boards. M. 1.25

BOURLET (C.) et BAUDOIN (P.). Cours abrégé de géométrie publié avec de nombreux exercices théoriques et pratiques et des applications au dessin géométrique. Paris, Hachette, 1906. 16mo. 8 + 408 pp. F. 2.50

] OZAL y OBEJERO (A.). Nociones de aritmética y geometría. Madrid, Arias, 1906. 12mo. 235 pp. P. 10.00

CAVEZZALI (A.). L'aritmetica e la geometria ad uso delle scuole elementari, con numerosi esercizi orali e scritti. Per la III classe. Milano, Abbiati, 1907. 16mo. 70 pp. L. 0.25

CIAMBERLINI (C.). Nozioni di aritmetica e geometria, con esercizi di calcolo mentale e scritto e problemi per la quarta classe elementare. Firenze, Bemporad, 1906. 16mo. 132 pp. (Biblioteca scolastica.) L. 0.50

COMBETTE (E.). Cours de trigonométrie. 4e édition, entièrement refondue. Paris, 1906. 8vo. Fr. 4.00

COSTANZO (G.) e NEGRO (C.). Geometria intuitiva e rudimenti di disegno geometrico per le prime tre classi del ginnasio. 2a edizione riveduta e corretta. Bologna, Zanichelli, 1907. 16mo. 168 pp.

DHYVERS. Questions et réponses sur les mathématiques et la physique. (Programmes du 27 juillet 1905.) Baccalauréat. Première partie (sections A et B). Paris, Croville-Morant [1906]. 16mo. 63 pp. Fr. 1.25

DREYFUS (L.). See FORT (L.).

DUPORT (J. B.). Lehrbuch der Arithmetik für die zweite und dritte Klasse der Mädchen-Lyzeen. 2te Auflage. Wien, Deuticke, 1906. 8vo.
Zweite Klasse. 3 + 83 pp. Boards. M. 1.20

——. Dritte Klasse. 3 + 96 pp. M. 1.25

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria. 2a edizione ridotta. Bologna, Zanichelli, 1907. 16mo. 279 pp. L. 2.50

FORT (L.) et DREYFUS (L.). Eléments de géométrie, conformes aux programmes du 27 juillet 1905, avec de nombreux exercices, figures et problèmes (classes de cinquième B et de quatrième A). Paris, Paulin, 1907. 16mo. 8 + 215 pp. Cloth. Fr. 2.50

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 91te Auflage. Halle, 1906. 8vo. 176 + 35 pp. Half leather. M. 2.50

JAHNE (J.) und BARBISCH (H.). Leitfaden der Geometrie und des geometrischen Zeichnens für Knabenbürgerschulen. 3te Stufe: Für die dritte Klasse. Mit einer Erklärungstafel für Bezirkspläne. 2te, verbesserte Auflage. Wien, Manz, 1906. 8vo. 4 + 80 pp. Boards. M. 0.90

KUHN (F.). Fragen und Aufgaben aus dem Anfangs-Kapitel der Planimetrie. München, Oldenbourg, 1906. 8vo. 7+48 pp. Boards. M. 0.80.

LEMBCKE (K.). Allgemeine Arithmetik in ihrer Beziehung zum praktischen Rechnen, für den Selbstunterricht insbesondere der Präparanden und Seminaraspiranten dargestellt. 3te, stereotypierte Auflage. Wismar, Hinstorff, 1906. 8vo. 8 + 107 pp. M. 2.00

MACÉ DE LÉPINAY (A.). See VACQUANT (C.).

MÜLLER (O.). Tavole di logaritmi con cinque decimali. 9a edizione, aumentata delle tavole dei logaritmi d'addizione e sottrazione per cura di M. Rajna. Milano, Hoepli, 1907. 16mo. 36 + 191 pp. (Manuali Hoepli.) L. 2.00

NEGRO (C.). See COSTANZO (G.).

RAJNA (M.). See MÜLLER (O.).

RIBONI (G.). Elementi di geometria, ad uso delle scuole secondarie superiori. 4a edizione, con aggiunte e correzioni. Bologna, Zanichelli, 1907. 16mo. 7 + 495 pp. L. 3.00

SCHAFHEITLIN (P.). Synthetische Geometrie der Kegelschnitte, für die Prima höherer Lehranstalten bearbeitet. Leipzig, Teubner, 1907. 8vo. 6 + 96 pp. Boards. M. 1.80

SCHIFFNER (F.). Vorschule der Geometrie. Ein Leitfaden für den Unterricht in der geometrischen Formenlehre an österreichischen Realschulen und verwandten Lehranstalten. Mit einem Anhang von 163 Uebungsaufgaben. Wien, Deuticke, 1906. 8vo. 48 pp. Boards. M. 1.00

SENDLER (R.). See BÖTTCHER (R.).

SPINELLI (D'A. A.). Regole di aritmetica pratica e nozioni di geometria intuitiva per la terza classe elementare maschile e femminile. Nuova edizione. Palermo, Biondo, 1906. 16mo. 63 pp. L. 0.45

——. Regole di aritmetica pratica e nozioni di geometria intuitiva per la quarta classe elementare maschile e femminile. Nuova edizione. Palermo, Biondo, 1906. 16mo. 124 pp. L. 0.60

VACQUANT (C.) et MACÉ DE LÉPINAY (A.). Eléments de géométrie à l'usage des classes de sciences (second cycle). Classes de deuxième, de première C et D et de mathématiques. 15e édition, conforme aux programmes du 27 juillet 1905. Paris, Masson, 1907. 16mo. 672 pp. Fr. 5.25

WIENECKE (E.). Die Grundlehren der Planimetrie in genetischer Darstellung, mit reichem Aufgabenmaterial, nebst einer Formentabelle des rechtwinkligen Dreiecks. Berlin, Röthig [1906]. 8vo. 8 + 174 + 11 pp. M. 2.50

WOLF (F. C.). Praktische Geometrie für den Schul- und Selbstunterricht. Nach den Grundsätzen der Anschauung und Konzentration in genetischer Stufenfolge aufgebaut und unter besonderer Berücksichtigung der praktischen Bedürfnisse bearbeiteter Lehrgang, mit dem ersten Preise gekrönt. Ausgabe für Lehrer. Mit Lösungen zu den Berechnungsaufgaben der Schülerhefte. 2te, verbesserte Auflage. Leipzig, Wunderlich, 1907. 8vo. 7 + 181 pp. M. 2.00

III. APPLIED MATHEMATICS.

ANDOYER (H.). Cours d'astronomie professé à la Faculté des sciences de Paris. Vol. I: Astronomie théorique. Paris, Hermann, 1906. 8vo. 222 pp. (Autogr.) Fr. 10.00

APPELL (P.) et CHAPPUIS (J.). Leçons de mécanique élémentaire, à l'usage des élèves des classes de mathématiques A et B, conformément aux programmes du 31 mai 1902. 2e édition, entièrement refondue. Paris, Gauthier-Villars, 1907. 16mo. 244 pp F. 3.00

BEZOLD (W. VON). Gesammelte Abhandlungen aus den Gebieten der Meteorologie und des Erdmagnetismus. In Gemeinschaft mit A. Coym herausgeben vom Verfasser. Braunschweig, Vieweg, 1906. 8vo. 8 + 448 pp., 3 plates. M. 14.00

BOWDITCH (N.). The American practical navigator, being an epitome of navigation and nautical astronomy. Revised edition. Washington, Superintendent of documents, 1906. 8vo. 652 pp. Cloth. $2.25

BRILLOUIN (M.). Leçons sur la viscosité des liquides et des gaz. (En 2 parties.) Partie 1: Généralités; viscosité des liquides. Paris, Gauthier-Villars, 1907. 8vo. 7 + 228 pp. Fr. 9.00

CHAPPUIS (J.). See APPELL (P.).

CHOLLET (T.) et MINEUR (P.). Traité de géométrie descriptive. Première partie à l'usage des élèves des classes de première C et D. 4e édition, conforme au programme du 27 juillet 1905. Paris, Vuibert, 1907. 8vo. 342 pp.

CRANDALL (C.). Text-book on geodesy and least squares; prepared for the use of civil engineering students. New York, Wiley, 1907. 8vo. 10 + 329 pp. Cloth. $3.00

DUHEM (P.). Recherches sur l'élasticité. De l'équilibre et du mouvement des milieux vitreux; les milieux vitreux peu déformés; la stabilité des milieux élastiques; propriétés générales des ondes dans les milieux visqueux et non visqueux. Paris, Gauthier-Villars, 1906. 4to. 218 pp. (Extrait des *Annales scientifiques de l'Ecole Normale supérieure* (3) 21, 22, 23.) Fr. 8.00

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